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Uniqueness of Admissible Solutions of the Riemann Problem for a System of Conservation Laws of Mixed Type

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1. INTRODUCTION

It is well known that the classical solution of the initial value problem for a strictly hyperbolic system of conservation laws

$$\frac{\partial u}{\partial t}+\frac{\partial f(u)}{\partial x}=0 \tag{1.1}$$

exists only locally in time, in general, and one has to consider a weak solution which satisfies the system (1.1) only in the distributional sense in order to obtain a globally defined solution which may contain discontinuities. However, weak solutions are not unique and one needs some admissibility criteria for choosing preferred solutions—admissible weak solutions which are physically reasonable. There are four kinds of commonly used admissibility criteria proposed from either physical or mathematical consideration, namely, the shock criterion (including Lax shock condition and Oleinik–Liu shock (E) condition), the viscosity criterion, the entropy criterion, and the entropy rate criterion [CH, D1, D2, LA1, LA2, LI1, LI2, SM].

Consider the model system of (1.1)

$$\begin{aligned} v_t+p(u)_x&=0\\ u_t-v_x&=0 \end{aligned} \tag{1.2}$$

describing the one dimensional isothermal motion of a compressible elastic fluid or solid in a Lagrangian coordinate system, where v denotes the

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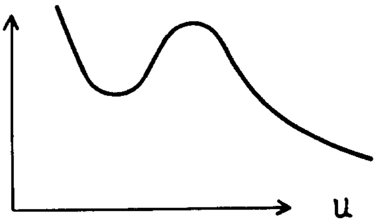

FIGURE 1.1

velocity, u the specific volume for a fluid or displacement gradient for a solid, and $-p$ is the stress which is determined through a constitutive relation to u.

For many materials, $p(u)$ is a decreasing function of u, and the system (1.2) is strictly hyperbolic. Moreover, when $p(u)$ is a concave function of u, (1.2) is genuinely nonlinear in which case it can be shown that the above four kinds of admissibility criteria are equivalent and they give the same unique admissible weak solution for the test Cauchy problem—the Riemann problem [D2]. When $p(u)$ is not a concave function of u (but $p(u)$ is not a linear function of u), (1.2) is not genuinely nonlinear, three kinds of the above criteria (shock (E), viscosity, and entropy rate) are equivalent which give the same unique admissible weak solution for the Riemann problem that satisfies the Lax criterion and the entropy criterion also [D2].

However, the system (1.2) can be of mixed type when it is used for dynamic elastic bar theory where the stress–deformation relation is not monotone [J] or for the dynamics of a material exhibiting change of phase such as in a van der Waals fluid [SL] (see Fig. 1.1). How do we define an admissible weak solution for the Riemann problem in the case when (1.2) is a mixed type system and prove the existence and uniqueness? It is obvious that we will fail to have the existence if we insist on using the original admissible criteria for a strictly hyperbolic system and one needs to make the generalization from the admissible criteria for using in a mixed type system. Certain efforts have been made. (See [J, K1, K2, HA1, HA2, HO, HS1, HSL, P, SE1, SE2, SL].)

We will introduce a different generalization in Section 2—the generalization of the shock (E) criterion—and reconsider the viscosity criterion. For each of these two criteria, there are two different types of statement for a strictly hyperbolic system which are equivalent. For instance, a discontinuity $(\sigma; u_+, v_+; u_-, v_-)$ is called admissible according to the shock (E) criterion if $\sigma = \sigma_i(u_-, u_+)$ such that

(I) $(u_+, v_+) \in H_i(u_-, v_-)$, $i=1$ or 2, $H_i(u_-, v_-)$ is the Hugoniot locus determined by the Rankine–Hugoniot condition with (u_-, v_-) given, namely,
$\sigma(v-v_-)=p(u)-p(u_-)$, $\sigma(u-u_-)=-(v-v_-)$; and
$\sigma_i(u_-, u_+) \leqslant \sigma_i(u_-, u)$ for all $(u, v) \in H_i(u_-, v_-)$ with u in between u_- and u_+.

An equivalent statement of (I) is

(II) $(u_-, v_-) \in H_i(u_+, v_+)$, $i=1$ or 2, $H_i(u_+, v_+)$ is the Hugoniot locus determined by the Rankine–Hugoniot condition with (u_+, v_+) given, namely,
$\sigma(v-v_+)=p(u)-p(u_+)$, $\sigma(u-u_+)=-(v-v_+)$; and
$\sigma_i(u, u_+) \leqslant \sigma_i(u_-, u_+)$ for all $(u, v) \in H_i(u_+, v_+)$ with u in between u_- and u_+.

(I) and (II) are equivalent since a discontinuity $(\sigma; u_+, v_+; u_-, v_-)$ is admissible according to (I) if and only if it is admissible according to (II) for system (1.2) when it is strictly hyperbolic.

Corresponding to the two types, we have two types ((I) and (II)) of generalization for both the shock (E) criterion and the viscosity criterion, stated in Section 2. We prove the existence and uniqueness of an admissible weak solution for the Riemann problem in Section 3 which satisfies the generalized shock (E) criterion type I. Moreover, we can show that the above admissible weak solution satisfies the generalized viscosity criterion type I, the generalized entropy criterion, and the entropy rate criterion.[1] Similarly, there exists a unique admissible weak solution for the Riemann problem which satisfies the generalized shock (E) criterion type II. This solution also satisfies the generalized viscosity criterion type II, the generalized entropy criterion.

Furthermore, for any given Riemann data it is shown in Section 4 that these solutions obtained by using the type I or the type II, respectively, are identical. This shows that the generalization of the shock (E) criterion introduced in this paper is a suitable admissibility criterion for the system (2.1) of mixed type.

2. Generalized Admissibility Criteria

For the system

$$\begin{aligned} v_t + p(u)_x &= 0 \\ u_t - v_x &= 0 \end{aligned} \tag{2.1}$$

[1] We will show that the admissible weak solution in this paper satisfies the entropy rate criterion in another paper [HS2] which concerns the system to nonisotermal motion.

the Rankine–Hugoniot condition takes the form

$$\begin{aligned} \sigma(v-v_-)-(p(u)-p(u_-))&=0 \\ \sigma(u-u_-)+(v-v_-)&=0 \end{aligned} \tag{2.2)$_\mathrm{I}$ $$

for any given state (u_-, v_-).

It can be shown that $(2.2)_\mathrm{I}$ defines two branches of continuous curves if $p'(u)<0$ for any $u>0$, denoted by $H_i(u_-, v_-)$, $i=1, 2$, on which the scalar function $\sigma=\sigma_i(u, v; u_-, v_-)$ can be defined respectively, namely,

$$\begin{aligned} \sigma_1 &= -\sqrt{-\frac{p(u)-p(u_-)}{u-u_-}} \quad \text{on} \quad H_1: \frac{v-v_-}{u-u_-}=\sqrt{-\frac{p(u)-p(u_-)}{u-u_-}} \\ \sigma_2 &= \sqrt{-\frac{p(u)-p(u_-)}{u-u_-}} \quad \text{on} \quad H_2: \frac{v-v_-}{u-u_-}=-\sqrt{-\frac{p(u)-p(u_-)}{u-u_-}}. \end{aligned} \qquad (2.3)_\mathrm{I}$$

Similarly, the Rankine–Hugoniot condition takes the form

$$\begin{aligned} \sigma(v-v_+)-(p(u)-p(u_+))&=0 \\ \sigma(u-u_+)+(v-v_+)&=0 \end{aligned} \qquad (2.2)_\mathrm{II}$$

for any given state (u_+, v_+).

Equation $(2.2)_\mathrm{II}$ defines two branches of continuous curves if $p'(u)<0$ for any $u>0$, denoted by $H_i(u_+, v_+)$, $i=1, 2$, on which the scalar function $\sigma=\sigma(u, v; u_+, v_+)$ can be defined respectively, namely,

$$\begin{aligned} \sigma_1 &= -\sqrt{-\frac{p(u)-p(u_+)}{u-u_+}} \quad \text{on} \quad H_1: \frac{v-v_+}{u-u_+}=\sqrt{-\frac{p(u)-p(u_+)}{u-u_+}} \\ \sigma_2 &= \sqrt{-\frac{p(u)-p(u_+)}{u-u_+}} \quad \text{on} \quad H_2: \frac{v-v_+}{u-u_+}=-\sqrt{-\frac{p(u)-p(u_+)}{u-u_+}}. \end{aligned} \qquad (2.3)_\mathrm{I}$$

A discontinuity $(\sigma; u_-, v_-; u_+, v_+)$ is said to be admissible according to the shock (E) criterion type I if

$$\begin{aligned} &(u_+, v_+)\in H_i(u_-, v_-), \qquad \sigma=\sigma_i(u_+, u_-), \quad i=1 \text{ or } 2 \\ &\sigma_i(u; u_-)\geqslant \sigma_i(u_+; u_-) \qquad \text{for all } u \text{ between } u_- \text{ and } u_+. \end{aligned} \qquad (2.4)_\mathrm{I}$$

A discontinuity $(\sigma; u_-, v_-; u_+, v_+)$ is said to be admissible according to the shock (E) criterion type II if

$$\begin{aligned} &(u_-, v_-)\in H_i(u_+, v_+), \qquad \sigma=\sigma_i(u_+, u_-), \quad i=1 \text{ or } 2 \\ &\sigma_i(u; u_+)\leqslant \sigma_i(u_-; u_+) \qquad \text{for all } u \text{ between } u_- \text{ and } u_+. \end{aligned} \qquad (2.4)_\mathrm{II}$$

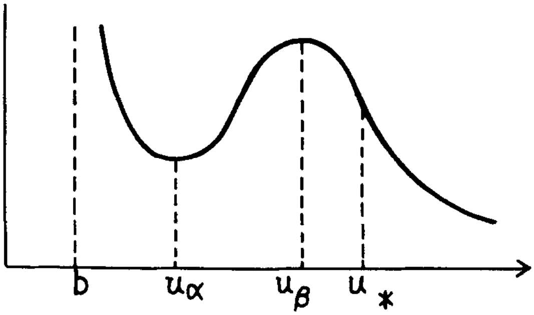

FIGURE 2.1

Now suppose that $p(u)$ has the type of graph as in Fig. 2.1, namely, $p(u)$ satisfies the following hypothesis (H) [HS1].

(H) (i) $p(u)$ is a smooth function defined on (b, ∞), where b is a given positive constant.

(ii) $p'(u)<0$ for $0<b<u<u_\alpha$ or $u>u_\beta$ and $p'(u_\alpha)=p'(u_\beta)=0$, $p'(u)\to 0$ as $u\to\infty$.

(iii) $p'(u)>0$ for $u_\alpha<u<u_\beta$.

For simplicity, we make further assumptions:

(iv) $p''(u)\geqslant 0$ if $u\leqslant u_\alpha$ and $p''(u)$ changes sign only once for $u>u_\beta$, where $u=u_*$ and $p(u_*)>p(u_\alpha)$ for definiteness.

(v) $\int_u^{u_0}\sqrt{-p'(\eta)}\,d\eta\to\infty$ and $u\to b$ for any given $u_0\leqslant u_\alpha$, $\int_{u_0}^{u}\sqrt{-p'(\eta)}\,d\eta\to\infty$ as $u\to+\infty$ for any given $u_0\geqslant u_\beta$.

The system (2.1) is a mixed type system then since the eigenvalues $\lambda_i(u)$ $(i=1,2)$ are complex-valued for $u_\alpha<u<u_\beta$.

It is shown in [HS1] that the Hugoniot locus $H_i(u_-, v_-)$ (or $H_i(u_+, v_+)$) is not necessarily a continuous curve any more and it can be disconnected.

DEFINITION 2.1. A discontinuity $(\sigma; u_-, v_-; u_+, v_+)$ is said to be admissible according to the generalized shock (E) criterion type I if

$$(u_+, v_+)\in H_i(u_-, v_-),\ \sigma=\sigma_i(u_+, u_-),\ i=1 \text{ or } 2\ \sigma_i(u;u_-)\geqslant \sigma_i(u_+;u_-) \text{ for any } u \text{ between } u_- \text{ and } u_+, \text{ where } \sigma_i(u;u_-) \text{ is defined.} \tag{2.5$_{\rm I}$}$$

DEFINITION 2.2. A discontinuity $(\sigma;\ u_-, v_-;\ u_+,\ v_+)$ is said to be admissible according to the generalized shock (E) criterion type II if

$(u_-, v_-) \in H_i(u_+, v_+)$, $\sigma = \sigma_i(u_+, u_-)$, $i = 1$ or 2 $\sigma_i(u; u_+) \leqslant \sigma_i(u_-; u_+)$ for any u between u_- and u_+, where $\sigma_i(u; u_+)$ is defined. $(2.5)_{\mathrm{II}}$

Consider the admissibility with respect to the viscosity criterion next. The associated viscosity system of (2.1) takes the form

$$\begin{aligned} v_t + p(u)_x &= \mu v_{xx} \\ u_t - v_x &= 0, \end{aligned} \tag{2.6}$$

where $\mu > 0$ is the assumed viscosity constant.

Let $(\sigma;\ u_-, v_-; u_+, v_+)$ be a discontinuity satisfying the Rankine–Hugoniot condition (2.2). This discontinuity is said to be admissible according to the viscosity criterion if the wave is a limit as $\mu \to 0^+$ of the traveling wave solution $u(x, t) = u((x - \sigma t)/\mu)$, $v(x, t) = v((x - \sigma t)/\mu)$ of the system (2.6) with the boundary condition

$$(u(-\infty), v(-\infty); u(+\infty), v(+\infty)) = (u_-, v_-; u_+, v_+). \tag{2.7}$$

For the traveling wave solution, the system (2.6) becomes

$$\begin{aligned} -\sigma v' &= (-p + v')' \\ -\sigma u' &= v', \end{aligned} \tag{2.8}$$

where $' = d/d\xi$, $\xi = (x - \sigma t)/\mu$. The integration of (2.8) from $-\infty$ to ξ coupled with (2.7) yields

$$\sigma u'(\xi) + \sigma^2(u(\xi) - u_-) + p(u(\xi)) - p(u_-) = 0. \tag{2.9}$$

For the hyperbolic system (2.1), $\sigma \neq 0$ (this is true if $p'(u) < 0$ for any $u > 0$). For (2.9) to have a continuous solution satisfying $u(-\infty) = u_-$ and $u(+\infty) = u_+$ we must have $u'(\xi) \geqslant 0$ if $u_- < u_+$ or $u'(\xi) \leqslant 0$ if $u_- > u_+$, namely,

$$\begin{aligned} &\frac{p(u) - p(u_-)}{u - u_-} \leqslant \frac{p(u_+) - p(u_-)}{u_+ - u_-} \qquad \text{for any } u \text{ between } u_- \text{ and } u_+ \text{ if } \sigma > 0 \\ &\frac{p(u) - p(u_-)}{u - u_-} \geqslant \frac{p(u_+) - p(u_-)}{u_+ - u_-} \qquad \text{for any } u \text{ between } u_- \text{ and } u_+ \text{ if } \sigma < 0. \end{aligned} \tag{2.10}$$

This is the viscosity admissibility criterion for a discontinuity $(\sigma,\ u_-, v_-; u_+, v_+)$ satisfying the Rankine–Hugoniot condition when the system (2.1) is hyperbolic.

When the system (2.1) is of mixed type, it is possible for σ to be zero from the Rankine–Hugoniot condition. In the case of $\sigma = 0$, if the chord connecting $(u_-, p(u_-))$ and $(u_+, p(u_+))$ does not cut the graph of $p(u)$, the condition (2.10) (just replace $\sigma > 0$ or $\sigma < 0$ by $\sigma \geqslant 0$ or $\sigma \leqslant 0$ in (2.10)) can guarantee that (2.9) has a continuous solution satisfying the boundary condition, therefore the given discontinuity is admissible according to the viscosity criterion. However, if the chord connecting $(u_-, p(u_-))$ and $(u_+, p(u_+))$ cuts the graph of $p(u)$, it is impossible for (2.9) to have a continuous solution satisfying the boundary condition $u(-\infty) = u_-$, $u(+\infty) = u_+$, therefore the discontinuity $(\sigma; u_-, v_-; u_+, v_+)$ with $\sigma = 0$, $v_+ = v_-$, $p(u_+) = p(u_-)$, $u_- < u_\alpha$, $u_+ > u_\beta$ or $u_+ < u_\alpha$, $u_- > u_\beta$ would not be admissible according to the viscosity criterion. On the other hand, if one allows discontinuous traveling waves then $u = u_-$, $\xi < 0$, $u = u_+$, $\xi > 0$ is a solution of (2.9). This implies the generalized viscosity criterion.

DEFINITION 2.3. A discontinuity $(\sigma; u_-, v_-; u_+, v_+)$ satisfying the Rankine–Hugoniot condition is admissible according to the generalized viscosity criterion if either (2.10) is satisfied, namely, $(2.10)_1$ holds for $\sigma \geqslant 0$ or $(2.10)_2$ holds for $\sigma \leqslant 0$, or $\sigma = 0$ (with $v_+ = v_-$, $p(u_+) = p(u_-)$) while u_- and u_+ are located in different phases. (Namely, either $u_- < u_\alpha$, $u_+ > u_\beta$ or $u_+ < u_\alpha$, $u_- > u_\beta$.)

The above generalized viscosity criterion is called type I since we can integrate (2.8) from ξ to $+\infty$ and get the viscosity criterion type II. In fact, we obtain

$$\sigma u'(\xi) + \sigma^2(u - u_+) + p(u) - p(u_+) = 0 \tag{2.11}$$

instead of (2.9).

For (2.11) to have a continuous solution satisfying $u(-\infty) = u_-$ and $u(+\infty) = u_+$, we must have

$$\begin{aligned} &\frac{p(u) - p(u_+)}{u - u_+} \geqslant \frac{p(u_-) - p(u_+)}{u_- - u_+} \qquad \text{for any } u \text{ between } u_- \text{ and } u_+ \text{ if } \sigma > 0 \\ &\frac{p(u) - p(u_+)}{u - u_+} \leqslant \frac{p(u_-) - p(u_+)}{u_- - u_+} \qquad \text{for any } u \text{ between } u_- \text{ and } u_+ \text{ if } \sigma < 0. \end{aligned} \tag{2.12}$$

This is the viscosity admissibility criterion for a discontinuity $(\sigma, u_-, v_-; u_+, v_+)$ satisfying the Rankine–Hugoniot condition when the system (2.1) is hyperbolic. From (2.12) we obtain the generalized viscosity criterion type II.

DEFINITION 2.4. A discontinuity $(\sigma; u_-, v_-; u_+, v_+)$ satisfying the Rankine–Hugoniot condition is admissible according to the generalized viscosity criterion type II if either $(2.12)_1$ holds for $\sigma \geqslant 0$ or $(2.12)_2$ holds for $\sigma \leqslant 0$, or $\sigma = 0$ (with $v_+ = v_-$, $p(u_+) = p(u_-)$), while u_- and u_+ are located in different phases.

The entropy criterion postulates that there is a nontrivial convex function of $\eta(u, v)$, so-called entropy, which, together with the entropy flux q, satisfies an additional conservation law

$$\eta_t + q_x = 0 \tag{2.13}$$

for the smooth solution (u, v) of (2.1). For the nonsmooth solution the entropy criterion asserts

$$\eta_t + q_x \leqslant 0. \tag{2.14}$$

The natural "entropy" for (2.1) is the total mechanical energy

$$\eta(u, v) = \tfrac{1}{2}v^2 - \int^u p(\xi)\, d\xi \tag{2.15}$$

which is a convex function for the case when the system (2.1) is hyperbolic ($p'(u) < 0$ for any $u > 0$). It is easy to check that η and $q(u, v) = vp(u)$ satisfy (2.13) for the smooth solution of (2.1). For a special nonsmooth solution which only contains one discontinuity $(\sigma; u_-, v_-; u_+, v_+)$ (it satisfies the Rankine–Hugoniot condition then), (2.14) becomes

$$\sigma \left\{ \frac{p(u_+) + p(u_-)}{2} (u_+ - u_-) - \int_{u_-}^{u_+} p(\xi)\, d\xi \right\} \geqslant 0, \tag{2.16}$$

which is the entropy criterion for the admissible discontinuity satisfying the Rankine–Hugoniot condition.

However, the function of $\eta(u, v)$ in (2.15) is no longer a convex function for the mixed type system (2.1) in which case we call (2.16) the generalized entropy criterion for the system (2.1).

3. THE EXISTENCE AND UNIQUENESS OF THE ADMISSIBLE WEAK SOLUTION FOR THE RIEMANN PROBLEM

Consider the system (2.1) with the Riemann initial condition

$$(u, v)|_{t=0} = \begin{cases} (u_-, v_-) & \text{for} \quad x < 0 \\ (u_+, v_+) & \text{for} \quad x > 0. \end{cases} \tag{3.1}$$

We look for self-similar solutions $u=u(\xi)$, $v=v(\xi)$, $\xi=x/t$ for the Riemann problem (2.1), (3.1) which is admissible according to the generalized shock (E) condition type I. We show the existence and uniqueness of the admissible weak solution and verify that this solution satisfies the other generalized admissibility criteria mentioned in Section 2. (For the viscosity criterion, it should be type I, of course). The corresponding results with the generalized showk (E) condition type II are also obtained.

Substitute $(u(\xi), v(\xi))$ into (2.1), and it follows that

$$\begin{pmatrix} \xi & -p'(u) \\ 1 & \xi \end{pmatrix} \begin{pmatrix} \dfrac{dv}{d\xi} \\ \dfrac{du}{d\xi} \end{pmatrix} = 0,$$

which supplies the solution wherever it is smooth. Namely, either $u=$ constant, $v=$ constant, called constant states, or $\xi=\lambda_i(u)$ and the vector $(dv/d\xi, du/d\xi)^T$ is parallel to the right eigenvector r_i, corresponding to λ_i, $i=1, 2$. This defines the ith rarefaction wave solution if $\lambda_i(u)$ is defined as a real-valued function and $\lambda_i(u)$ is monotone along the integral curve of the vector field r_i, the so-called rarefaction wave curve, denoted by R_i. More precisely, for $u \leqslant u_\alpha$ or $u \geqslant u_\beta$,

$$\lambda_1 = -\sqrt{-p'(u)} \qquad (\text{or } \lambda_2 = \sqrt{-p'(u)})$$

$$R_1 \,(\text{or } R_2) \text{ is the integral curve of } \frac{dv}{du} = \sqrt{-p'(u)}$$

$$\left(\text{or } \frac{dv}{du} = -\sqrt{-p'(u)}\right).$$

Suppose that $p(u)$ satisfies the hypotheses (H) throughout the paper. It is easy to show the following proposition about rarefaction waves.

PROPOSITION 3.1 [HS1]. *For any given (u_0, v_0) with $b<u_0<u_\alpha$, the state (u, v) which can be joined to (u_0, v_0) on the right hand side (namely, in the direction of ξ increasing) by a 1th (or 2th) rarefaction wave is defined by*

$$\begin{cases} u_0 \leqslant u \leqslant u_\alpha \\ v - v_0 = \displaystyle\int_{u_0}^{u} \sqrt{-p'(\eta)}\, d\eta \end{cases} \qquad \left(\text{or} \begin{cases} b < u \leqslant u_0 \\ v - v_0 = -\displaystyle\int_{u_0}^{u} \sqrt{-p'(\eta)}\, d\eta \end{cases} \right) \qquad (3.2)_1$$

denoted still by R_1 (*or* R_2), *and for any given* (u_0, v_0) *with* $u_\beta < u_0 \leqslant u_*$, *the state* (u, v) *which can be joined to* (u_0, v_0) *on the right hand side by a* 1*th* (*or* 2*th*) *rarefaction wave is defined by*

$$\begin{cases} u_\beta \leqslant u \leqslant u_0 \\ v - v_0 = \int_{u_0}^{u} \sqrt{-p'(\eta)}\, d\eta \end{cases} \qquad \left(or \begin{cases} u_0 \leqslant u \leqslant u_* \\ v - v_0 = -\int_{u_0}^{u} \sqrt{-p'(\eta)}\, d\eta \end{cases} \right), \tag{3.2$_2$}$$

while for any given (u_0, v_0) *with* $u_* < u_0 < \infty$, *the above kinds of states* (u, v) *are defined by*

$$\begin{cases} u \geqslant u_0 \\ v - v_0 = \int_{u_0}^{u} \sqrt{-p'(\eta)}\, d\eta \end{cases} \qquad \left(or \begin{cases} u_* \leqslant u \leqslant u_0 \\ v - v_0 = -\int_{u_0}^{u} \sqrt{-p'(\eta)}\, d\eta \end{cases} \right). \tag{3.2$_3$}$$

PROPOSITION 3.2. *For any given* (u_0, v_0) *with* $b < u_0 \leqslant u_\alpha$, *the state* (u, v) *which can be joined to* (u_0, v_0) *on the left hand side, namely, in the direction of* ξ *decreasing, by a* 1*th* (*or* 2*th*) *rarefaction wave is defined by*

$$\begin{cases} b < u \leqslant u_0 \\ v - v_0 = \int_{u_0}^{u} \sqrt{-p'(\eta)}\, d\eta \end{cases} \qquad \left(or \begin{cases} u_0 \leqslant u \leqslant u_\alpha \\ v - v_0 = -\int_{u_0}^{u} \sqrt{-p'(\eta)}\, d\eta \end{cases} \right) \tag{3.3$_1$}$$

denoted by $R_1^{(\mathrm{II})}$ (*or* R_2^{II}) *as distinct from the* R_1 (*or* R_2) *defined in Proposition* 3.1. *For any given* (u_0, v_0) *with* $u_\beta \leqslant u_0 \leqslant u_*$, *the state* (u, v) *which can be joined to* (u_0, v_0) *on the left hand side by a* 1*th* (*or* 2*th*) *rarefaction wave is defined by*

$$\begin{cases} u_0 \leqslant u \leqslant u_* \\ v - v_0 = \int_{u_0}^{u} \sqrt{-p'(\eta)}\, d\eta \end{cases} \qquad \left(or \begin{cases} u_\beta \leqslant u \leqslant u_0 \\ v - v_0 = -\int_{u_0}^{u} \sqrt{-p'(\eta)}\, d\eta \end{cases} \right). \tag{3.3$_2$}$$

For any given (u_0, v_0) *with* $u_* \leqslant u_0 < \infty$, *the above kinds of states* (u, v) *are defined by*

$$\begin{cases} u_* \leqslant u \leqslant u_0 \\ v - v_0 = \int_{u_0}^{u} \sqrt{-p'(\eta)}\, d\eta \end{cases} \qquad \left(or \begin{cases} u_0 \leqslant u < \infty \\ v - v_0 = -\int_{u_0}^{u} \sqrt{-p'(\eta)}\, d\eta \end{cases} \right). \tag{3.3$_3$}$$

Turn to discontinuity now. For any given (u_-, v_-), *we consider the state* (u_+, v_+) *which belongs to* $H_i(u_-, v_-)$ *(see* $(2.3)_{\mathrm{I}}$*) and supplies, together with* (u_-, v_-), *an admissible discontinuity according to Definition* 2.1. *We denote the set of these states by* $S_i(u_-, v_-)$, $i = 1, 2$.

PROPOSITION 3.3 [HS1]. *Corresponding to different locations of* (u_-, v_-) *with* $b < u_- \leqslant \bar{u}$, $\bar{u} < u_- < u_\alpha$, $u_\alpha \leqslant u_- \leqslant u_\beta$, $u_\beta < u_- < u_*$, $u_* \leqslant u_- \leqslant \hat{u}$, $\hat{u} < u_- < \infty$, $S_i(u_-, v_-)$ *can be defined respectively as follows. (*$\hat{u}$ *is defined by* $p(\hat{u}) = p(u_\alpha)$; $\bar{u}$ *is defined by* $p(\bar{u}) = p(u_\beta)$.)

For the case when $b < u_- \leqslant \bar{u}$,

$$S_1 : \begin{cases} b < u \leqslant u_- \\ (u, v) \in H_1(u_-, v_-) \end{cases}$$

$$S_2 : \begin{cases} u_- \leqslant u \leqslant u_{R_\beta(-)} \qquad or \qquad u \geqslant u_{D_\beta(-)} \\ (u, v) \in H_2(u_-, v_-), \end{cases}$$

where $u_{R_\beta(-)}$ *is defined by*

$$\frac{p(u_{R_\beta}) - p(u_-)}{u_{R_\beta} - u_-} = p'(u_{R_\beta}), \qquad u_{R_\beta} \geqslant u_\beta \tag{3.4}$$

and $u_{D_\beta(-)}$ *is defined by*

$$\frac{p(u_{D_\beta}) - p(u_-)}{u_{D_\beta} - u_-} = \frac{p(u_{R_\beta}) - p(u_-)}{u_{R_\beta} - u_-} = p'(u_{R_\beta}), \qquad u_{D_\beta} > u_{R_\beta}. \tag{3.5}$$

For the case when $\bar{u} < u_- < u_\alpha$, *there are two subcases:* $u_- < \tilde{u}$ *or* $u_- > \tilde{u}$, *where* $\tilde{u}$ *is defined by*

$$p'(\tilde{u}) = p'(\tilde{\tilde{u}}) = \frac{p(\tilde{u}) - p(\tilde{\tilde{u}})}{\tilde{u} - \tilde{\tilde{u}}}, \qquad \tilde{u} < u_\alpha, \tilde{\tilde{u}} > u_\beta. \tag{3.6}$$

When $\bar{u} < u_- < \tilde{u}$,

$$S_1 : \begin{cases} b < u \leqslant u_- \\ (u, v) \in H_1(u_-, v_-) \end{cases}$$

$$S_2 : \begin{cases} u_- \leqslant u \leqslant u_{B_e(-)} \qquad or \qquad u = u_{B_\beta(-)} \\ (u, v) \in H_2(u_-, v_-), \end{cases}$$

where

$$p(u_{B_e}) = p(u_-) = p(u_{B_\beta}) \qquad and \qquad p'(u_{B_e}) > 0, \qquad p'(u_{B_\beta}) \leqslant 0, u_{B_\beta} \geqslant u_\beta. \tag{3.7}$$

When $\tilde{u} < u_- < u_\alpha$,

$$S_1:\begin{cases} b < u \leqslant u_- \quad \text{or} \quad u_{L\beta(-)} \leqslant u \leqslant u_{R\beta(-)} \\ (u, v) \in H_1(u_-, v_-) \end{cases}$$

$$S_2:\begin{cases} u_- \leqslant u \leqslant u_{B_e(-)} \quad \text{or} \quad u = u_{B\beta(-)} \\ (u, v) \in H_2(u_-, v_-), \end{cases}$$

where $u_{L\beta(-)}$ *is defined by*

$$p'(u_-) = \frac{p(u_{L_\beta}) - p(u_-)}{u_{L_\beta} - u}, \qquad u_{L_\beta} \geqslant u_\beta. \tag{3.8}$$

For the case when $u_\alpha \leqslant u_- \leqslant u_\beta$,

$$S_1:\begin{cases} b < u \leqslant u_{B_\alpha(-)} \quad \text{or} \quad u_{B\beta(-)} \leqslant u \leqslant u_{R\beta(-)} \\ (u, v) \in H_1(u_-, v_-) \end{cases}$$

$$S_2:\begin{cases} u = u_{B_\alpha(-)} \quad \text{or} \quad u = u_{B\beta(-)} \\ (u, v) \in H_2(u_-, v_-), \end{cases}$$

where $u_{B_\alpha(-)}$ *is defined by*

$$p(u_{B_\alpha}) = p(u_-) \qquad \text{and} \qquad u_{B_\alpha} \leqslant u_\alpha. \tag{3.9}$$

For the case when $u_\beta < u_- < u_*$,

$$S_1:\begin{cases} b < u \leqslant u_{L_\alpha(-)} \quad \text{or} \quad u_- \leqslant u \leqslant u_{R\beta(-)} \\ (u, v) \in H_1(u_-, v_-) \end{cases}$$

$$S_2:\begin{cases} u_{B_e(-)} \leqslant u \leqslant u_- \quad \text{or} \quad u = u_{B_\alpha(-)} \quad \text{or} \quad u_{L\beta(-)} \leqslant u < \infty \\ (u, v) \in H_2(u_-, v_-), \end{cases}$$

where $u_{L_\alpha(-)}$ *is defined by*

$$\frac{p(u_{L_\alpha}) - p(u_-)}{u_{L_\alpha} - u_-} = p'(u_-), \qquad u_{L_\alpha} \leqslant u_\alpha. \tag{3.10}$$

For the case when $u_* \leqslant u_- \leqslant \hat{u}$,

$$S_1:\begin{cases} u_{R\beta(-)} \leqslant u \leqslant u_- \quad \text{or} \quad u \leqslant u_{D_\alpha(-)} \\ (u, v) \in H_1(u_-, v_-) \end{cases}$$

$$S_2:\begin{cases} u \geqslant u_- \quad \text{or} \quad u_{B_e(-)} \leqslant u \leqslant u_{L_e(-)} \quad \text{or} \quad u = u_{B_\alpha(-)} \\ (u, v) \in H_2(u_-, v_-), \end{cases}$$

where $u_{D_\alpha(-)}$ is defined by

$$\frac{p(u_{D_\alpha})-p(u_-)}{u_{D_\alpha}-u_-}=\frac{p(u_{R_\beta})-p(u_-)}{u_{R_\beta}-u_-}=p'(u_{R_\beta}), \qquad u_{D_\alpha}<u_\alpha \tag{3.11}$$

and $u_{L_e(-)}$ is defined by

$$\frac{p(u_{L_e})-p(u_-)}{u_{L_e}-u_-}=p'(u_-), \qquad \tilde{u}\leqslant u_{L_e}\leqslant u_*. \tag{3.12}$$

For the case when $u_->\hat{u}$, there are two subcases: $u_-<\tilde{\tilde{u}}$ or $u_-\geqslant\tilde{\tilde{u}}$. When $\hat{u}<u_-<\tilde{\tilde{u}}$,

$$S_1:\begin{cases} u_{R_\beta(-)}\leqslant u\leqslant u_- \qquad \text{or} \qquad u\leqslant u_{D_\alpha}(-) \\ (u,v)\in H_1(u_-,v_-) \end{cases}$$

$$S_2:\begin{cases} u_{R_\alpha(-)}\leqslant u\leqslant u_{L_e(-)} \qquad \text{or} \qquad u_-\leqslant u<\infty \\ (u,v)\in H_2(u_-,v_-), \end{cases}$$

where $u_{R_\alpha(-)}$ is defined by

$$\frac{p(u_{R_\alpha})-p(u_-)}{u_{R_\alpha}-u_-}=p'(u_{R_\alpha}), \qquad u_{R_\alpha}\leqslant u_\alpha, \tag{3.13}$$

and when $u_-\geqslant\tilde{\tilde{u}}$,

$$S_1:\begin{cases} u_{R_\beta(-)}\leqslant u\leqslant u_- \qquad \text{or} \qquad u\leqslant u_{D_\alpha(-)} \\ (u,v)\in H_1(u_-,v_-) \end{cases}$$

$$S_2:\begin{cases} u\geqslant u_- \\ (u,v)\in H_2(u_-,v_-). \end{cases}$$

For any given (u_+,v_+), we consider the state (u_-,v_-) which belongs to $H_i(u_+,v_+)$ (see $(2.3)_{\text{II}}$) and supplies, together with (u_+,v_+), an admissible discontinuity according to Definition 2.2. We denote the set of these states by $S_i^{(\text{II})}(u_+,v_+)$, $i=1,2$, as distinct from the S_1 (or S_2) defined by Proposition 3.3.

PROPOSITION 3.4. *Corresponding to different locations of (u_+,v_+) with $b<u_+\leqslant u'_*$, $u'_*<u_+\leqslant\bar{u}$, $\bar{u}<u_+<u_\alpha$, $u_\alpha\leqslant u_+\leqslant u_\beta$, $u_\beta<u_+<u_*$, $u_*\leqslant u_+\leqslant\tilde{\tilde{u}}$, $\tilde{\tilde{u}}<u_+<\infty$, $S_i^{(\text{II})}(u_+,v_+)$ can be defined respectively ($i=1,2$) as follows, where u'_* is defined by*

$$\frac{p(u'_*)-p(u_*)}{u'_*-u_*}=p'(u_*).$$

For the case when $b < u_+ \leqslant u'_*$,

$$S_1^{(\mathrm{II})}: \begin{cases} u_+ < u < \infty \\ (u, v) \in H_1(u_+, v_+) \end{cases}$$

$$S_2^{(\mathrm{II})}: \begin{cases} b < u < u_+ \\ (u, v) \in H_2(u_+, v_+). \end{cases}$$

For the case when $u'_* < u_+ < \bar{u}$,

$$S_1^{(\mathrm{II})}: \begin{cases} u_+ < u < u_{L\beta(+)} \qquad or \qquad u_{D\beta(+)} < u < \infty \\ (u, v) \in H_1(u_+, v_+) \end{cases}$$

$$S_2^{(\mathrm{II})}: \begin{cases} b < u < u_+ \\ (u, v) \in H_2(u_+, v_+), \end{cases}$$

where $u_{L\beta(+)}$ *is defined by*

$$\frac{p(u_{L\beta(+)}) - p(u_+)}{u_{L\beta(+)} - u_+} = p'(u_{L\beta(+)}), \qquad u_{L\beta(+)} \geqslant u_\beta \tag{3.14}$$

and $u_{D\beta(+)}$ *is defined by*

$$\begin{aligned} \frac{p(u_{D\beta(+)}) - p(u_+)}{u_{D\beta(+)} - u_+} &= \frac{p(u_{L\beta(+)}) - p(u_+)}{u_{L\beta(+)} - u_+} \\ &= p'(u_{L\beta(+)}), \qquad u_{D\beta(+)} > u_{L\beta(+)}. \end{aligned} \tag{3.15}$$

For the case when $\bar{u} \leqslant u_+ < u_\alpha$ *there are two subcases:* $\bar{u} \leqslant u_+ \leqslant \tilde{u}$ *or* $\tilde{u} < u_+ < u_\alpha$. *When* $\bar{u} \leqslant u_+ \leqslant \tilde{u}$,

$$S_1^{(\mathrm{II})}: \begin{cases} u_+ < u < u_{B_\epsilon(+)} \qquad or \qquad u = u_{B\beta(+)} \\ (u, v) \in H_1(u_+, v_+) \end{cases}$$

$$S_2^{(\mathrm{II})}: \begin{cases} b < u < u_+ \\ (u, v) \in H_2(u_+, v_+) \end{cases}$$

and when $\tilde{u} < u_+ < u_\alpha$,

$$S_1^{(\mathrm{II})}: \begin{cases} u_+ < u < u_{B_\epsilon(+)} \qquad or \qquad u = u_{B\beta(+)} \\ (u, v) \in H_1(u_+, v_+) \end{cases}$$

$$S_2^{(\mathrm{II})}: \begin{cases} b < u < u_+ \qquad or \qquad u_{R\beta(+)} \leqslant u \leqslant u_{L\beta(+)} \\ (u, v) \in H_2(u_+, v_+), \end{cases}$$

where $u_{B_e(+)}$ *and* $u_{B_\beta(+)}$ *are defined by* (3.7), *replacing* u_- *by* u_+; $u_{R_\beta(+)}$ is *defined by*

$$\frac{p(u_{R_\beta(+)})-p(u_+)}{u_{R_\beta(+)}-u_+}=p'(u_+), \qquad u_{R_\beta(+)}\geqslant u_\beta. \tag{3.16}$$

For the case when $u_\alpha \leqslant u_+ \leqslant u_\beta$,

$$S_1^{(\mathrm{II})}: \begin{cases} u=u_{B_\beta(+)} \quad \text{or} \quad u=u_{B_\alpha(+)} \\ (u,v)\in H_1(u_+,v_+) \end{cases}$$

$$S_2^{(\mathrm{II})}: \begin{cases} b<u\leqslant u_{B_\alpha(+)} \quad \text{or} \quad u_{B_\beta(+)}\leqslant u\leqslant u_{L_\beta(+)} \\ (u,v)\in H_2(u_+,v_+), \end{cases}$$

where $u_{B_\alpha(+)}$ *is defined by* (3.9), *replacing* u *by* u_+.

For the case when $u_\beta < u_+ < u_*$,

$$S_1^{(\mathrm{II})}: \begin{cases} u_{B_e(+)}\leqslant u<u_+ \quad \text{or} \quad u=u_{B_\alpha(+)} \quad \text{or} \quad u_{R_\beta(+)}\leqslant u<\infty \\ (u,v)\in H_1(u_+,v_+) \end{cases}$$

$$S_2^{(\mathrm{II})}: \begin{cases} b<u\leqslant u_{B_\alpha(+)} \quad \text{or} \quad u_+<u\leqslant u_{L_\beta(+)} \\ (u,v)\in H_2(u_+,v_+). \end{cases}$$

For the case when $u_* \leqslant u_+ \leqslant \tilde{\tilde{u}}$,

$$S_1^{\mathrm{II})}: \begin{cases} u=u_{L_e(+)}\leqslant u\leqslant u_{R_e(+)} \quad \text{or} \quad u_+<u<\infty \\ (u,v)\in H_1(u_+,v_+) \end{cases}$$

$$S_2^{(\mathrm{II})}: \begin{cases} b<u\leqslant u_{D_\alpha(+)} \quad \text{or} \quad u_{L_\beta(+)}\leqslant u<u_+ \\ (u,v)\in H_2(u_+,v_+), \end{cases}$$

where u_{L_e} *is defined by*

$$\frac{p(u_{L_e(+)})-p(u_+)}{u_{L_e(+)}-u_+}=p'(u_{L_e}), \qquad \tilde{u}\leqslant u_{L_e(+)}<u_\beta; \tag{3.17}$$

$u_{R_e(+)}$ *is defined by*

$$\frac{p(u_{R_e(+)})-p(u_+)}{u_{R_e(+)}-u_+}=p'(u_+), \qquad \tilde{u}\leqslant u_{R_e(+)}<u_* \tag{3.18}$$

while $u_{D_\alpha(+)}$ *is defined by*

$$\begin{aligned} \frac{p(u_{D_\alpha(+)})-p(u_+)}{u_{D_\alpha(+)}-u_+}&=\frac{p(u_{L_\beta(+)})-p(u_+)}{u_{L_\beta(+)}-u_+} \\ &=p'(u_{L_\beta(+)}), \qquad u_{D_\alpha(+)}<u_\alpha. \end{aligned} \tag{3.19}$$

For the case when $\tilde{u} < u_+ < \infty$,

$$S_1^{(\mathrm{II})}: \begin{cases} u_+ < u < \infty \\ (u, v) \in H_1(u_+, v_+) \end{cases}$$

$$S_2^{(\mathrm{II})}: \begin{cases} b < u \leqslant u_{D_\alpha(+)} \quad \text{or} \quad u_{L_\beta(+)} \leqslant u < u_+ \\ (u, v) \in H_2(u_+, v_+). \end{cases}$$

DEFINITION 3.5. A single-valued function $(u(\xi), v(\xi))$ is called an admissible type I weak solution of (2.1), (3.1) if

I. It satisfies the boundary condition $(u, v) \to (u_\mp, v_\mp)$ as $\xi \to \mp\infty$.

II. It is either a rarefaction wave or a constant state wherever it is smooth.

III. Any discontinuity is admissible type I satisfying Definition 2.1.

DEFINITION 3.6. A single-valued function $(u(\xi), v(\xi))$ is called an admissible type II weak solution of (2.1), (3.1) if the items I and II in Definition 3.5 hold and item III is replaced by the following III. Any discontinuity is admissible type II satisfying Definition 2.2.

We will prove the existence and uniqueness of the admissible type I weak solution first. For convenience we will neglect the word type I in the discussion.

For any given (u_-, v_-), consider the set of all states which can be joined to (u_-, v_-), on the right hand side, by a single-valued function $(u(\xi), v(\xi))$, consisting of the first kind of waves. Namely, it contains either a 1-admissible discontinuity or a 1-rarefaction wave or a fan of such first kind waves. We denote this set by $\bar{W}_1(u_-, v_-)$, which is a curve on the (u, v) plane for given (u_-, v_-) but not necessarily connected. For each point $(u_1, v_1) \in \bar{W}_1(u_-, v_-)$, we determine the set of all states which can be joined to (u_1, v_1), on the right hand side, by a single-valued function $(u(\xi), v(\xi))$ consisting of the second kind of waves. Namely, it contains either a 2-admissible discontinuity or a 2-rarefaction wave or a fan of such second kind waves. We denote this set by $\bar{W}_2(u_1, v_1)$.

It has been proved [HS1] that, corresponding to different locations of (u_-, v_-), $\bar{W}_2(u_1, v_1)$ is a continuous curve, defined for $b < u < \infty$ and expressed by formula for any $(u_1, v_1) \in \bar{W}_1(u_-, v_1)$. For any given (u_-, v_-) we show next that the family of curves $\{\bar{W}_2(u_1, v_1), (u_1, v_1) \in \bar{W}_1(u_-, v_-)\}$ covers the whole domain D: $\{-\infty < v < \infty, b < u < \infty\}$ univalently, which implies the existence and uniqueness for the Riemann problem (2.1), (3.1).

For any $(u_+, v_+) \in D$, it can be easily shown, by using the formulas defining the curves $\bar{W}_2(u_1, v_1)$ and $\bar{W}_1(u_-, v_-)$, given in [HS1], that there

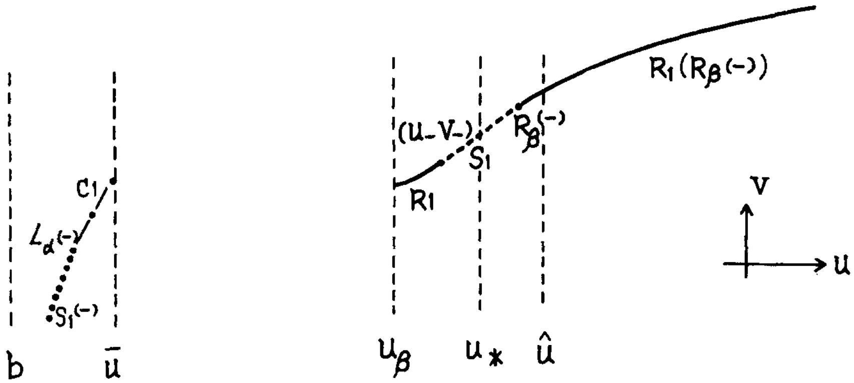

FIG. 3.1. The heavy solid lines denote curves R (R_1 or R_2). The dashed lines denote curves S (S_1 or S_2). The dotted lines denote curves C (C_1 or C_2). The vertical dashed lines denote the lines of $u = \text{const}$.

is a state (u_1, v_1) such that $(u_+, v_+) \in \bar{W}_2(u_1, v_1)$ and $(u_1, v_1) \in \bar{W}_1(u_-, v_-)$. This implies the existence of the Riemann problem. Now we are going to prove that for any $(u_+, v_+) \in D$, there is only one state (u_1, v_1) such that $(u_+, v_+) \in \bar{W}_2(u_1, v_1)$ and $(u_1, v_1) \in \bar{W}_1(u_-, v_-)$ which supplies a single-valued function $(u(\xi), v(\xi))$ that is an admissible solution of (2.1), (3.1). This gives the uniqueness of the Riemann problem. For definiteness, let us assume $u_\beta < u_- < u_*$, and the other cases can be treated similarly.

$\bar{W}_1(u_-, v_-)$ is defined by the formula (see Fig. 3.1)

$$\bar{W}_1(u_-, v_-) = \begin{cases} S_1(u_-, v_-) & \text{for} \quad b < u \leqslant u_{L_\alpha(-)} \\ C_1(u_\beta, u_-; R_1(-)) & \text{for} \quad u_{L_\alpha(-)} < u < \bar{u} \\ R_1(u_-, v_-) & \text{for} \quad u_\beta \leqslant u \leqslant u_- \\ S_1(u_-, v_-) & \text{for} \quad u_- < u \leqslant u_{R_\beta(-)} \\ R_1(u_{R_\beta(-)}, v_{R_\beta(-)}) & \text{for} \quad u_{R_\beta(-)} < u < \infty, \end{cases} \tag{3.20}$$

where $C_1(u_\beta, u_-; R_1(-))$ consists of states $(u_{L_\alpha(1)}, v_{L_\alpha(1)})$ such that corresponding to each state $(u_1, v_1) \in R_1(u_-, v_-)$ with $u_\beta \leqslant u_1 \leqslant u_-$, it holds that

$$\begin{gathered} \text{(3.10), replacing } u_- \text{ by } u_1 \\ (u_{L_\alpha(1)}, v_{L_\alpha(1)}) \in H_1(u_1, v_1) \end{gathered} \tag{3.21}$$

$(u_{L_\alpha(-)}, v_{L_\alpha(-)}) = (u_{L_\alpha(1)}, v_{L_\alpha(1)})|_{(u_1, v_1) = (u_-, v_-)}$ then, denoted by $L_\alpha(-)$.

$(u_{R_\beta(-)}, v_{R_\beta(-)} \in H_1(u_-, v_-))$ denoted by $R_\beta(-)$ and $u_{R_\beta(-)}$ is defined by (3.4).

It can be shown that the curve $\bar{W}_2(u_1, v_1)$ is defined as follows corresponding to different locations of (u_1, v_1) on $\bar{W}_1(u_-, v_-)$. (See Fig. 3.2.)

For any $(u_1, v_1) \in \bar{W}_1(u_-, v_-)$ with $b < u_1 \leqslant u'_*$ (u'_* is defined in Proposition 3.4),

$$\bar{W}_2(u_1, v_1) = \begin{cases} R_2(u_1, v_1 & \text{for} \quad b < u \leqslant u_1 \\ S_2(u_1, v_1) & \text{for} \quad u_1 < u < \infty. \end{cases} \tag{3.22}$$

For any $(u_1, v_1) \in \bar{W}_1(u_-, v_-)$ with $u'_* < u_1 < \bar{u}$, there are $u_{R_\beta(1)}$ and $u_{D_\beta(1)}$, defined by (3.4) and (3.5), respectively, replacing u_- by u_1 there, and $\bar{W}_2(u_1, v_1)$ is defined as

$$\bar{W}_2(u_1, v_1) = \begin{cases} R_2(u_1, v_1) & \text{for} \quad b < u \leqslant u_1 \\ S_2(u_1, v_1) & \text{for} \quad u_1 < u \leqslant u_{R_\beta(1)} \\ R_2(u_{R_\beta(1)}, v_{R_\beta(1)}) & \text{for} \quad u_{R_\beta(1)} < u \leqslant u_* \\ C_2(u_{R_\beta(1)}, u_*;\, R_2(u_{R_\beta(1)}, v_{R_\beta(1)}) & \text{for} \quad u_* < u < u_{D_\beta(1)} \\ S_2(u_1, v_1) & \text{for} \quad u_{D_\beta(1)} \leqslant u < \infty, \end{cases} \tag{3.23}$$

where $R_\beta(1) = ((u_{R_\beta(1)}, v_{R_\beta(1)}) \in S_2(u_1, v_1))$, $D_\beta(1) = (u_{D_\beta(1)}, v_{D_\beta(1)} \in S_2(u_1, v_1))$, and $C_2(u_{R_\beta(1)}, u_*;\, R_2(u_{R_\beta(1)}, v_{R_\beta(1)})$ consists of states $(u_{L_\beta(2)}, v_{L_\beta(2)})$ such that corresponding to each state $(u_2, v_2) \in R_2(u_{R_\beta(1)}, v_{R_\beta(1)})$ with $u_{R_\beta(1)} \leqslant u_2 \leqslant u_*$ it holds that

$$\begin{gathered} \text{(3.8), replacing } u_- \text{ by } u_2 \\ (u_{L_\beta(2)}, v_{L_\beta(2)}) \in H_2(u_2, v_2). \end{gathered} \tag{3.24}$$

$u_{L_\beta(2)}$ varies from u_* to $u_{D_\beta(1)}$ as u_2 varies from u_* to $u_{R_\beta(1)}$, and $(u_{L_\beta(2)}, v_{L_\beta(2)})$ and (u_2, v_2) supply an admissible second kind of discontinuity. Moreover, $(u_{L_\beta(2)}, v_{L_\beta(2)}) \in S_2(u_1, v_1)$ when $u_2 = u_{R_\beta(1)}$ and $\bar{W}_2(u_1, v_1)$ is a smooth curve defined for $b < u < \infty$ on which $v \to +\infty$ as $u \to b$ and $v \to -\infty$ as $u \to +\infty$. Particularly, when $u_1 = \bar{u}$,

$$\bar{W}_2(u_-, v_-) = \begin{cases} R_2(u_1, v_1) & \text{for} \quad b < u \leqslant u_1 = \bar{u} \\ S_2(u_1, v_1) & \text{for} \quad \bar{u} < u \leqslant u_{R_\beta(1)} = u_\beta \\ R_2(u_{R_\beta(1)}, v_{R_\beta(1)}) & \text{for} \quad u_{R_\beta(1)} < u \leqslant u_* \\ C_2(u_{R_\beta(1)}, u_*;\, R_2(u_{R_\beta(1)}, v_{R_\beta(1)}) & \text{for} \quad u_* < u < \infty. \end{cases} \tag{3.25}$$

Now we turn to (u_1, v_1) on $\bar{W}_1(u_-, v_-)$ for $u_1 \geqslant u_\beta$. For any $(u_1, v_1) \in \bar{W}_1(u_-, v_-)$ with $u_\beta < u_1 < u_*$, there are $u_{L_\beta(1)}$, $u_{B_e(1)}$, and $u_{B_\alpha(1)}$, defined in (3.8), (3.7), and (3.9), respectively, replacing u_- by u_1. $\bar{W}_2(u_1, v_1)$ is defined as

$$\bar{W}_2(u_1, v_1) = \begin{cases} R_2(u_{B_\alpha(1)}, v_{B_\alpha(1)}) & \text{for } b < u < u_{B_\alpha(1)} \\ S_2(u_{B_\alpha(1)}, v_{B_\alpha(1)}) & \text{for } u_{B_\alpha(1)} \leqslant u < u_{B_e(1)} \\ S_2(u_1, v_1) & \text{for } u_{B_e(1)} \leqslant u < u_1 \\ R_2(u_1, v_1) & \text{for } u_1 \leqslant u \leqslant u_* \\ C_2(u_1, u_*; R_2(u_1, v_1)) & \text{for } u_* < u < u_{L_\beta(1)} \\ S_2(u_1, v_1) & \text{for } u_{L_\beta(1)} \leqslant u < \infty, \end{cases} \tag{3.26}$$

where $B_\alpha(1) = (u_{B_\alpha(1)}, v_{B_\alpha(1)}) \in S_2(u_1, v_1)$ with $\sigma_2(u_{B_\alpha(1)}, v_{B_\alpha(1)}; u_1, v_1) = 0$, $B_e(1) = (u_{B_e(1)}, v_{B_e(1)}) \in S_2(u_1, v_1)$ with $\sigma_2(u_{B_e(1)}, v_{B_e(1)}; u_1, v_1) = 0$, and $C_2(u_1, u_*; R_2(u_1, v_1))$ consists of states $(u_{L_\beta(2)}, v_{L_\beta(2)})$ such that corresponding to each state $(u_2, v_2) \in R_2(u_1, v_1)$ with $u_1 \leqslant u_2 \leqslant u_*$, (3.24) holds. Moreover, $(u_{L_\beta(2)}, v_{L_\beta(2)}) \in S_2(u_1, v_1)$ when $u_2 = u_1$, denoted by $L_\beta(1) = (u_{L_\beta(1)}, v_{L_\beta(1)})$. When $u_1 = u_\beta$, $(u_{B_e(1)}, v_{B_e(1)})$ coincide with (u_1, v_1), $u_{L_\beta(1)} \to \infty$, and $\bar{W}_2(u_1, v_1)$ is defined as

$$\bar{W}_2(u_1, v_1) = \begin{cases} R_2(u_{B_\alpha(1)}, v_{B_\alpha(1)}) & \text{for } b < u < u_{B_\alpha(1)} = \bar{u} \\ S_2(u_{B_\alpha(1)}, v_{B_\alpha(1)}) & \text{for } u_{B_\alpha(1)} \leqslant u < u_1 = u_\beta \\ R_2(u_1, v_1) & \text{for } u_\beta \leqslant u \leqslant u_* \\ C_2(u_1, u_*; R_2(u_1, v_1)) & \text{for } u_* < u < \infty, \end{cases} \tag{3.27}$$

which defines the same curve as defined by (3.25).

For any $(u_1, v_1) \in \bar{W}_1(u_-, v_-)$ with $u_* < u_1 < \hat{u}$, by replacing u_- by u_1, one defines $u_{L_e(1)}$, $u_{B_e(1)}$, and $u_{B_\alpha(1)}$ in (3.12), (3.7), and (3.9), respectively, and $\bar{W}_2(u_1, v_1)$ is defined as

$$\bar{W}_2(u_1, v_1) = \begin{cases} R_2(u_{B_\alpha(1)}, v_{B_\alpha(1)}) & \text{for } b < u < u_{B_\alpha(1)} \\ S_2(u_{B_\alpha(1)}, v_{B_\alpha(1)}) & \text{for } u_{B_\alpha(1)} \leqslant u < u_{B_e(1)} \\ S_2(u_1, v_1) & \text{for } u_{B_e(1)} \leqslant u < u_{L_e(1)} \\ C_2(u_*, u_1; R_2(u_1, v_1)) & \text{for } u_{L_e(1)} < u < u_* \\ R_2(u_1, v_1) & \text{for } u_* \leqslant u \leqslant u_1 \\ S_2(u_1, v_1) & \text{for } u_1 < u < \infty, \end{cases} \tag{3.28}$$

where $L_e(1) = (u_{L_e(1)}, v_{L_e(1)}) \in S_2(u_1, v_1)$.

For any $(u_1, v_1) \in \bar{W}_1(u_-, v_-)$ with $\hat{u} < u_1 < \tilde{\tilde{u}}$, there are $u_{R_\alpha(1)}$ and $L_e(1)$ defined in (3.13) and (3.12), respectively, replacing u_- by $u_1 \cdot \bar{W}_2(u_1, v_1)$ is defined as

$$\bar{W}_2(u_1, v_1) = \begin{cases} R_2(u_{R_\alpha(1)}, v_{R_\alpha(1)}) & \text{for} \quad b < u < u_{R_\alpha(1)} \\ S_2(u_1, v_1) & \text{for} \quad u_{R_\alpha(1)} \leqslant u < u_{L_e(1)} \\ C_2(u_*, u_1; R_2(u_1, v_1)) & \text{for} \quad u_{L_e(1)} < u < u_* \\ R_2(u_1, v_1) & \text{for} \quad u_* \leqslant u \leqslant u_1 \\ S_2(u_1, v_1) & \text{for} \quad u_1 < u < \infty, \end{cases} \tag{3.29}$$

where $(u_{L_e(1)}, v_{L_e(1)})$ and $C_2(u_*, u_1; R_2(u_1, v_1))$ are defined in the same way as before, and $R_\alpha(1) = (u_{R_\alpha(1)}, v_{R_\alpha(1)}) \in S_2(u_1, v_1)$.

For any $(u_1, v_1) \in \bar{W}_1(u_-, v_-)$ with $\tilde{\tilde{u}} < u_1 < \infty$, $\bar{W}_2(u_1, v_1)$ is defined as

$$\bar{W}_2(u_1, v_1) = \begin{cases} R_2(\tilde{u}, \tilde{v}) & \text{for} \quad b < u \leqslant \tilde{u} \\ C_2(u_*, \tilde{\tilde{u}}; R_2(u_1, v_1)) & \text{for} \quad \tilde{u} < u < u_* \\ R_2(u_1, v_1) & \text{for} \quad u_* \leqslant u \leqslant u_1 \\ S_2(u_1, v_1) & \text{for} \quad u_1 < u < \infty, \end{cases} \tag{3.30}$$

where $(\tilde{u}, \tilde{v}) \in S_2(\tilde{\tilde{u}}, \tilde{\tilde{v}})$, $(\tilde{\tilde{u}}, \tilde{\tilde{v}}) \in R_2(u_1, v_1)$, and $\tilde{u}$ and $\tilde{\tilde{u}}$ are defined in Proposition 3.3.

The family of the curves $\{\bar{W}_2(u_1, v_1), (u_1, v_1) \in \bar{W}_1(u_-, v_-)\}$ is shown in Fig. 3.2, which covers the domain D. This implies the existence for the Riemann problem (2.1), (3.1).

We show next that no two of the curves corresponding to different $(u_1, v_1) \in \bar{W}_1(u_-, v_-)$ will intersect each other in the domain D, which implies the uniqueness. For proving it, we need the following lemmas.

LEMMA 3.7. *If $u_A = u_B$, then $H_i(u_A, v_A)$ and $H_i(u_B, v_B)$ are parallel wherever they are defined, $i = 1$ or 2.*

LEMMA 3.8. *Suppose that $H_2(u_0, v_0)(H_1(u_0, v_0))$ is defined for $u_0 \leqslant u \leqslant u_A$, $(u_1, v_1) \in H_2(u_0, v_0)(H_1(u_0, v_0))$ with $u_0 < u_1 < u_A$, and $H_2(u_1, v_1)(H_1(u_1, v_1))$ is defined for $u_1 \leqslant u \leqslant u_A$. Then $H_2(u_1, v_1)(H_1(u_1, v_1))$ is located above (below) $H_2(u_0, v_0)(H_1(u_0, v_0))$ for $u_1 < u \leqslant u_A$.*

LEMMA 3.9. *Suppose that $H_2(u_0, v_0)(H_1(u_0, v_0))$ is defined for $u_B \leqslant u \leqslant u_0$, $(u_1, v_1) \in H_2(u_0, v_0)(H_1(u_0, v_0))$ with $u_B < u_1 < u_0$, and $H_2(u_1, v_1)(H_1(u_1, v_1))$ is defined for $u_B \leqslant u \leqslant u_1$, then $H_2(u_1, v_1)(H_1(u_1, v_1))$ is located below (above) $H_2(u_0, v_0)(H_1(u_0, v_0))$ for $u_B \leqslant u < u_1$.*

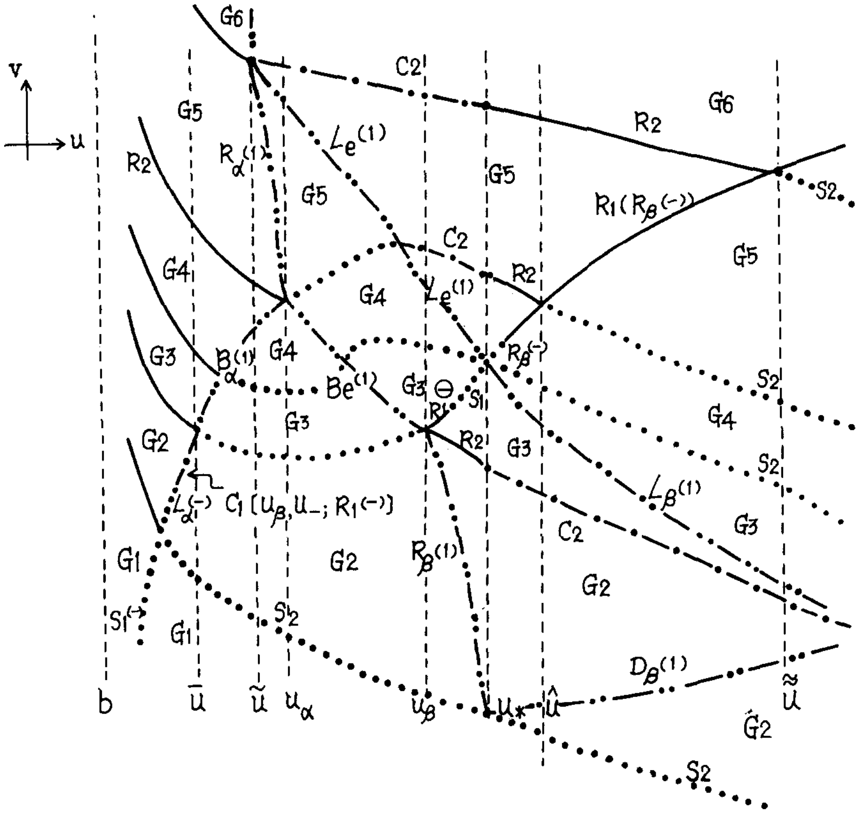

FIG. 3.2. The double-dotted lines denote other curves, such as $R_\alpha(1)$, $R_\beta(1)$, $L_\alpha(1)$, $L_\beta(1)$, $L_e(1)$, $B_\alpha(1)$, $B_e(1)$, $B_\beta(1)$.

LEMMA 3.10. *For any given* (u_0, v_0), *either* $b < u_0 < u_\alpha$ *or* $u_\beta < u_0 < \infty$, *there is an interval containing* u_0, *say* $u_B < u_0 < u_A$, *on which both* $R_i(u_0, v_0)$ *and* $H_i(u_0, v_0)$ *are defined*, $i = 1$ *or* 2. *Furthermore*, $H_2(u_0, v_0)(H_1(u_0, v_0))$ *is located below (above)* $R_2(u_0, v_0)(R_1(u_0, v_0))$ *for* $u > u_0$ *and above (below)* $R_2(u_0, v_0)(R_1(u_0, v_0))$ *for* $u < u_0$ *on the interval.*

Lemma 3.7 follows from (2.3) immediately. The proof of Lemma 3.9 is similar to Lemma 3.8. We give the proofs of Lemma 3.8 and Lemma 3.10 next.

Proof of Lemma 3.8. In view of (2.3),

$$
\begin{aligned}
v\,|_{H_2(u_1, v_1)} - v\,|_{H_2(u_0, v_0)} &= \sqrt{-[p(u)-p(u_0)](u-u_0)} - \sqrt{-[p(u)-p(u_1)](u-u_1)} \\
&\quad - \sqrt{-[p(u_1)-p(u_0)](u_1-u_0)}
\end{aligned}
$$

for $u_1 \leqslant u \leqslant u_A$ denoted by $f(u)$. It is clear that $f(u) \geqslant 0$ means

$$\sqrt{-[p(u)-p(u_0)](u-u_0)} \geqslant \sqrt{-[p(u)-p(u_1)](u-u_1)} + \sqrt{-[p(u_1)-p(u_0)](u_1-u_0)},$$

namely,

$$\sqrt{-\{p(u)-p(u_1)+p(u_1)-p(u_0)\}(u-u_1+u_1-u_0)} \geqslant \sqrt{-[p(u)-p(u_1)](u-u_1)} + \sqrt{-[p(u_1)-p(u_0)](u_1-u_0)},$$

which is equivalent to

$$-[p(u)-p(u_1)](u_1-u_0) - [p(u_1)-p(u_0)](u-u_1) \geqslant 2\sqrt{\{-[p(u)-p(u_1)](u-u_1)\}\{-[(p(u_1)-p(u_0)](u_1-u_0)\}},$$

namely,

$$\{\sqrt{-[p(u)-p(u_1)](u_1-u_0)} - \sqrt{-[p(u_1)-p(u_0)](u-u_1)}\}^2 \geqslant 0$$

since both $-[p(u)-p(u_1)](u_1-u_0)$ and $[p(u_1)-p(u_0)](u-u_1)$ are positive. This finishes the proof of Lemma 3.8.

Proof of Lemma 3.10. In view of (2.3) and (3.2), the first part of the lemma is easily obtained. For proving the second part, let us consider $u_A > u > u_0$ first.

$$\begin{aligned}
& v\,|_{R_2(u_0,\, v_0)} - v\,|_{H_2(u_0,\, v_0)} \\
&= -\int_{u_0}^{u} \sqrt{-p'(\eta)}\, d\eta + \sqrt{-[p(u)-p(u_0)](u-u_0)} \\
&= -\left\{\int_{u_0}^{u} \sqrt{-p'(\eta)}\, d\eta - \sqrt{-[p(u)-p(u_0)](u-u_0)}\right\} \\
&\geqslant -\left\{\sqrt{\int_{u_0}^{u} -p'(\eta)\, d\eta}\, \sqrt{\int_{u_0}^{u} 1 \cdot d\eta} - \sqrt{-[p(u)-p(u_0)](u-u_0)}\right\} \\
&= 0.
\end{aligned}$$

Similarly, $v\,|_{R_2(u_0,\, v_0)} - v\,|_{H_2(u_0,\, v_0)} \leqslant$ for $u_B < u < u_0$.

The family of curves $\bar{W}_2(u_1, v_1)$ is divided into six groups $G_1, ..., G_6$, and the boundary curves of different groups are shown in Fig. 3.2. In order to prove the uniqueness, it suffices to show that any two curves $\bar{W}_2(u_1, v_1)$ in the same group, including the boundary curves, do not intersect each other.

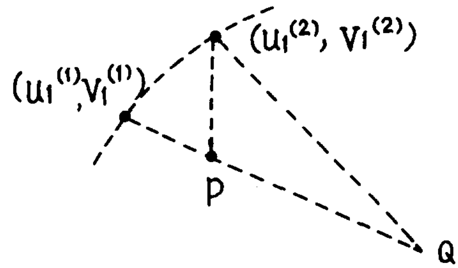

FIGURE 3.3

1. Curves $\bar{W}_2(u_1, v_1)$ in G_1:

Let $\bar{W}_2(u_1^{(1)}, v_1^{(1)})$ and $\bar{W}_2(u_1^{(2)}, v_1^{(2)})$ be two curves in G_1. They cannot intersect each other on the left side of the curve $\bar{W}_1(u_-, v_-)$ since they are curves R_2 and cannot intersect each other on the right side of the curve $\bar{W}_1(u_-, v_-)$ either due to Lemma 3.7 and Lemma 3.8. In fact, let $u_1^{(1)} < u_1^{(2)}$, and $\bar{W}_2(u_1^{(1)}, v_1^{(1)})$ and $\bar{W}_2(u_1^{(2)}, v_1^{(2)})$ intersect at a point Q (see Fig. 3.3). Observe that $\bar{W}_2(u_1^{(1)}, v_1^{(1)})$ and $\bar{W}_2(u_1^{(2)}, v_1^{(2)})$ are curves $S_2(u_1^{(1)}, v_1^{(1)})$ and $S_2(u_1^{(2)}, v_1^{(2)})$, respectively, on the right side of the curve $\bar{W}_1(u_-, v_-)$. Draw line $u = u_1^{(2)}$, which intersects to $\bar{W}_2(u_1^{(1)}, v_1^{(1)})$ at p. Since $u_p = u_1^{(2)}$, $S_2(p)$ is parallel to $S_2(u_1^{(2)}, v_1^{(2)})$ inview of Lemma 3.7. On the other hand, p is located on $S_2(u_1^{(1)}, v_1^{(1)})$ with $u_p > u_1^{(1)}$, therefore $S_2(p)$ is located above $S_2(u_1^{(1)}, v_1^{(1)})$ for $u > u_p$ in view of Lemma 3.8. This gives the contradiction.

2. Curves $\bar{W}_2(u_1, v_1)$ in G_2:

Let $\bar{W}_2(u_1^{(1)}, v_1^{(1)})$ and $\bar{W}_2(u_1^{(2)}, v_1^{(2)})$ be two curves in G_2 with $u_1^{(1)} < u_1^{(2)}$. They cannot intersect each other in the part $R_2(u_1, v_1)$, $R_2(u_{R_\beta(1)}, v_{R_\beta(1)})$, or $S_2(u_1, v_1)$ for the same reason as in case 1. Let us consider the part $C_2(u_{R_\beta(1)}, u_*;\ R_2(u_{R_\beta(1)}, v_{R_\beta(1)}))$. Suppose $\bar{W}_2(u_1^{(1)}, v_1^{(1)})$ and $W_2(u_1^{(2)}, v_1^{(2)})$ intersect each other in this part. Namely, there is a point M_1 on $R_2(u_{R_\beta(1^{(1)})}, v_{R_\beta(1^{(1)})})$ and a point M_2 on $R_2(u_{R_\beta(1^{(2)})}, v_{R_\beta(1^{(2)})})$ such that $S_2(M_1)$ intersects $S_2(M_2)$ at Q and

$$\frac{p(u_{M_1}) - p(u_Q)}{u_{M_1} - u_Q} = p'(u_{M_1}), \qquad \frac{p(u_{M_2}) - p(u_Q)}{u_{M_2} - u_Q} = p'(u_{M_2}).$$

This implies that $u_{M_1} = u_{M_2}$ since $u_Q > u_*$, u_{M_1}, $u_{M_2} \in [u_\beta, u_*]$, therefore $S_2(M_1)$ and $S_2(M_2)$ are parallel by Lemma 3.7, which gives the contradiction.

By similar argument with the help of Lemmas 3.7–3.10, we can deal with the other cases. In summary, we have proved

THEOREM 3.11. *For any given initial data $(u_\mp, v_\mp)$ in the domain D,* *there is one and only one admissible weak solution, satisfying Definition* 3.5, *for the Riemann problem* (2.1), (3.1).

Next we are going to show that our admissible type I weak solution obtained in Theorem 3.11 satisfies the generalized viscosity criterion type I and the generalized entropy criterion. In doing this we only need to show that our admissible type I discontinuity satisfies the generalized viscosity criterion type I and the generalized entropy criterion for discontinuity. Consider each case in Proposition 3.3, and it is easy to verify that for any $(u_+, v_+) \in S_1(u_-, v_-)$ with $\sigma_1 < 0$, it holds that

$$\frac{p(u) - p(u_-)}{u - u_-} \geqslant \frac{p(u_+) - p(u_-)}{u_+ - u_-}$$

for any u between u_- and u_+, while for any $(u_+, v_+) \in S_2(u_-, v_-)$ with $\sigma_2 > 0$, it holds that

$$\frac{p(u) - p(u_-)}{u - u_-} \leqslant \frac{p(u_+) - p(u_-)}{u_+ - u_-}$$

for any u between u_- and u_+. Moreover, for any $(u_+, v_+) \in S_i(u_-, v_-)$ with $\sigma_i = 0$, it holds that $v_+ = v_-$, $p(u_+) = p(u_-)$. Therefore, any admissible discontinuity according to our generalized shock (E) criterion type I satisfies the generalized viscosity criterion type I.

Now turn to the generalized entropy criterion. Consider each case in Proposition 3.3, and it can be easily verified that for any $(u_+, v_+) \in S_1(u_-, v_-)$ with $\sigma_1 < 0$, it holds that

$$\frac{p(u_+) + p(u_-)}{2} (u_+ - u_-) \leqslant \int_{u_-}^{u_+} p(\xi)\, d\xi,$$

while for any $(u_+, v_+) \in S_2(u_-, v_-)$ with $\sigma_2 > 0$, it holds that

$$\frac{p(u_+) + p(u_-)}{2} (u_+ - u_-) \geqslant \int_{u_-}^{u_+} p(\xi)\, d\xi.$$

Therefore, any admissible discontinuity according to our generalized shock (E) criterion type I satisfies the generalized entropy criterion

$$\sigma \left\{ \frac{p(u_+) + p(u_-)}{2} (u_+ - u_-) - \int_{u_-}^{u_+} p(\xi)\, d\xi \right\} \geqslant 0.$$

Thus, we end up with

THEOREM 3.12. *The admissible weak solution for the Riemann problem*

(2.1), (3.1) *obtained in Theorem* 3.11 *satisfies the generalized viscosity criterion* (type I) *and the generalized entropy criterion also.*

Similarly, we can prove the existence and uniqueness of the admissible type II weak solution. For any given (u_+, v_+), we consider the set of all states which can be joined to (u_+, v_+), on the left hand side, by a single-valued function $(u(\xi), v(\xi))$, consisting of the second kind of waves. Namely, it contains either a 2-admissible type II discontinuity or a 2-rarefaction wave or a fan of such second kind waves. We denote this set by $\bar{W}_2^{(\mathrm{II})}(u_+, v_+)$, which is a curve on the (u, v) plane for given (u_+, v_+) but not necessarily connected. For each point $(u_2, v_2) \in \bar{W}_2^{(\mathrm{II})}(u_+, v_+)$, we determine the set of all states which can be joined to (u_2, v_2), on the left hand side, by a single-valued function $(u(\xi), v(\xi))$ consisting of the first kind of waves. Namely, it contains either a 1-admissible type II discontinuity or a 1-rarefaction wave or a fan of such first kind waves. We denote this set by $\bar{W}_1^{(\mathrm{II})}(u_2, v_2)$. Corresponding to different locations of (u_+, v_+), it can be proved that $\bar{W}_1^{(\mathrm{II})}(u_2, v_2)$ is a continuous curve, defined for $b < u < \infty$ and expressed by precise formula for any $(u_2, v_2) \in \bar{W}_2^{(\mathrm{II})}(u_+, v_+)$, in a similar way as $(u_1, v_1) \in \bar{W}_1(u_-, v_-)$ for $\bar{W}_2(u_1, v_1)$.

For any given (u_+, v_+), we can show, similarly as for the admissible type I, that the family of curves $\{\bar{W}_1^{(\mathrm{II})}(u_2, v_2),\ (u_2, v_2) \in \bar{W}_2^{(\mathrm{II})}(u_+, v_+)\}$ covers the whole domain D univalently, which implies the existence and uniqueness for the Riemann problem (2.1), (3.1) with Definition 3.6.

THEOREM 3.13. *For any given initial data* $(u_\mp, v_\mp)$ *in the domain* D, *there is one and only one admissible weak solution, satisfying Definition* 3.6, *for the Riemann problem* (2.1), (3.1).

Finally, we can show that our admissible type II discontinuity satisfies the generalized viscosity criterion type II and the generalized entropy criterion. Therefore, we end up with

THEOREM 3.14. *The admissible weak solution for the Riemann problem* (2.1), (3.1), *obtained in Theorem* 3.13, *satisfies the generalized viscosity criterion type* II *and the generalized entropy criterion also.*

Remark 3.15. We have made the convention that the solutions having the same figure on the (x, t) plane are identified as the same solution no matter whether or not their images on the phase plane are the same.

4. THE IDENTIFICATION OF TYPE I AND TYPE II SOLUTIONS

In this section we will show that for given Riemann data the unique admissible type I solution obtained in Theorem 3.11 is the same as the unique admissible type II solution obtained in Theorem 3.13. This shows that the generalized shock (E) criterion is a suitable admissibility criterion for system (1.2) of mixed type.

It is shown in Section 3 that for any given (u_-, v_-) the family of curves $\{W_2(u_1, v_1),\ (u_1, v_1) \in W_1(u_-, v_-)\}$ covers the whole domain D: $\{-\infty < v < \infty,\ b < u < \infty\}$ univalently which determines the unique type I weak solution for the Riemann problem (2.1), (3.1). Namely, for any $(u_+, v_+) \in D$, there is one and only one type I weak solution $(u(\xi), v(\xi))$ $(\xi = x/t)$ satisfying the boundary conditions $(u(\xi), v(\xi)) \to (u_\mp, v_\mp)$ as $\xi \to \mp\infty$. We will show next that for this state $(u_+, v_+) \in D$, we are able to construct a type II solution which satisfies the same boundary conditions and is of the same configuration as the corresponding type I solution. Due to the uniqueness of the type II weak solution, it follows that the type I weak solution and type II weak solution for this pair of Riemann data are identical. This shows the identification of type I and type II weak solutions for any given Riemann data because of the arbitrariness of (u_-, v_-), (u_+, v_+).

Let us consider the case when $u_\beta < u_- < u_*$ now; the other cases can be treated similarly. For any given (u_-, v_-) with $u_\beta < u_- < u_*$, the family of the curves $\{W_2(u_1, v_1),\ (u_1, v_1) \in W_1(u_-, v_-)\}$ is shown in Fig. 3.2, which is divided into six groups $G_1, \ldots, G_6$. Corresponding to different locations of (u_+, v_+), the configuration of the type I solution is different.

Case 1. $(u_+, v_+) \in G_1$.

When (u_+, v_+) is located on the right of the curve $W_1(u_-, v_-)$, the corresponding type I solution contains a 1-shock, joining (u_-, v_-) and (u_1, v_1), and a 2-shock, joining (u_1, v_1) and (u_+, v_+), as shown in Fig. 4.1, where $(u_+, v_+) \in S_2(u_1, v_1)$ while $(u_1, v_1) \in S_1(u_-, v_-)$ with $u_1 < u'_*$. Now, starting from (u_+, v_+), we construct $S_2^{(\mathrm{II})}(u_+, v_+)$. Due to Proposition 3.4, $(u_1, v_1) \in S_2^{(\mathrm{II})}(u_+, v_+)$ always. Starting from (u_1, v_1) we construct $S_1^{(\mathrm{II})}(u_1, v_1)$; and it is obvious, since $b < u_1 < u'_*$, that $(u_-, v_-) \in$

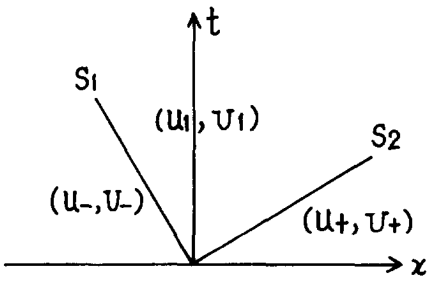

FIGURE 4.1

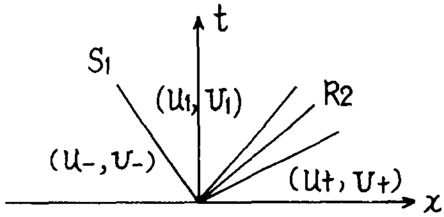

FIG. 4.2. The single ray denotes shock wave S. The fan of rays denotes rarefaction wave R.

$S_1^{(\mathrm{II})}(u_1, v_1)$. Thus, we have constructed a type II solution which has the same configuration as shown in Fig. 4.1.

When (u_+, v_+) is located on the left of the curve $W_1(u_-, v_-)$, the corresponding type I solution contains a 1-shock, joining (u_-, v_-) and (u_1, v_1), and a 2-rarefaction wave, joining (u_1, v_1) and (u_+, v_+), as shown in Fig. 4.2. Obviously, $(u_1, v_1) \in R_2^{(\mathrm{II})}(u_+, v_+)$ and $(u_-, v_-) \in S_1^{(\mathrm{II})}(u_1, v_1)$. Therefore, we obtain a type II solution which is the same as shown in Fig. 4.2.

Case 2. $(u_+, v_+) \in G_2$.

When (u_+, v_+) is located on the right of the curve $D_\beta(1)$ (see Fig. 3.2), there are two kinds of wave patterns for the type I solution: either

$$(u_-, v_-) \xrightarrow{\quad R_1 \quad} (\bar{u}_1, \bar{v}_1) \xrightarrow[\text{with } \sigma_1 = \lambda_1(\bar{u}_1)]{S_1} (u_1, v_1) \xrightarrow{\quad S_2 \quad} (u_+, v_+),$$

as shown in Fig. 4.3, or

$$(u_-, v_-) \xrightarrow{\quad S_1 \quad} (u_1, v_1) \xrightarrow{\quad S_2 \quad} (u_+, v_+),$$

as shown in Fig. 4.4, where $u_1 = u_{L_\alpha(\bar{u}_1, \bar{v}_1)}$, satisfying $u_{L_\alpha(u_-, v_-)} < u_{L_\alpha(\bar{u}_1, \bar{v}_1)} < \bar{u}$ for the first case while $u'_* < u_1 \leqslant u_{L_\alpha(u_-, v_-)}$ for the second case.

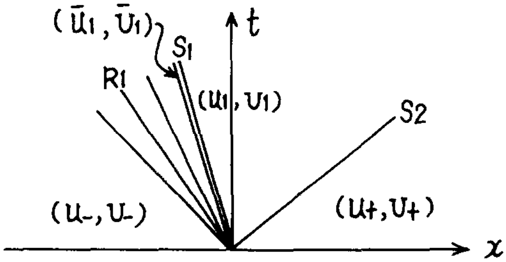

FIG. 4.3. The set of double rays denotes contact discontinuity S.

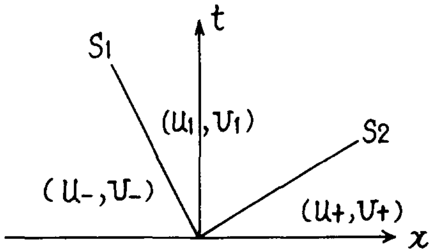

FIGURE 4.4

No matter which cases, $(u_1, v_1) \in S_2^{(\mathrm{II})}(u_+, v_+)$ since (u_+, v_+) is located on the right of the curve $D_\beta(1)$. Starting from (u_1, v_1), we construct $S_1^{(\mathrm{II})}(u_1, v_1)$. It is easy to see that $(\bar{u}_1, \bar{v}_1) \in S_1^{(\mathrm{II})}(u_1, v_1)$ with $\sigma_1 = \lambda_1(\bar{u}_1)$ for the first case while $(u_-, v_-) \in S_1^{(\mathrm{II})}(u_1, v_1)$ for the second case. Therefore, we have constructed the corresponding type II solution, the same as shown in Figs. 4.3 and 4.4, respectively.

When (u_+, v_+) is locted between the curves $u = u_*$ and $D_\beta(1)$ (see Fig. 3.2), there are two kinds of wave patterns for the type I solution also: either

$$(u_-, v_-) \xrightarrow{\quad R_1 \quad} (\bar{u}_1, \bar{v}_1) \xrightarrow[\text{with } \sigma_1 = \lambda_1(\bar{u}_1)]{S_1} (u_1, v_1)$$

$$\xrightarrow[\text{with } \sigma_2 = \lambda_2(R_\beta(I))]{S_2} R_\beta(1) \xrightarrow{\quad R_2 \quad} (u_2, v_2) \xrightarrow[\text{with } \sigma_2 = \lambda_2(u_2)]{S_2} (u_+, v_+),$$

as shown in Fig. 4.5, or

$$(u_-, v_-) \xrightarrow{\quad S_1 \quad} (u_1, v_1) \xrightarrow[\text{with } \sigma_2 = \lambda_2(R_\beta(1))]{S_2}$$

$$(R_\beta(1)) \xrightarrow{\quad R_2 \quad} (u_2, v_2) \xrightarrow[\text{with } \sigma_2 = \lambda_2(u_2)]{S_2} (u_+, v_+),$$

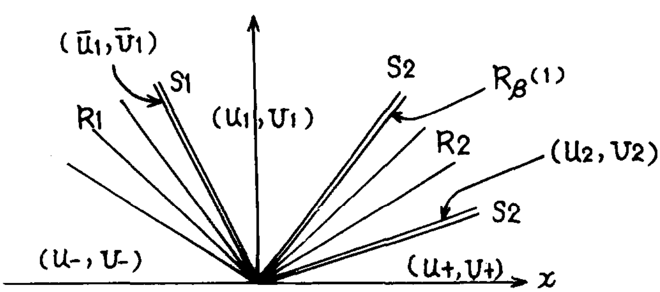

FIGURE 4.5

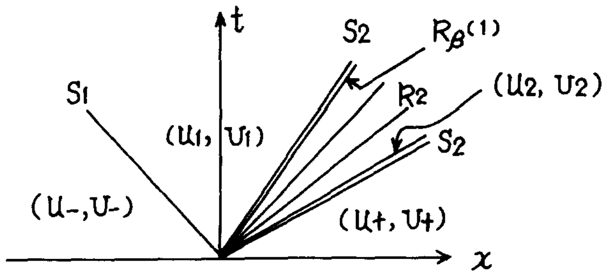

FIGURE 4.6

as shown in Fig. 4.6, where $(u_+, v_+) = L_\beta(u_2, v_2)$, $(u_+, v_+) = L_\beta(u_2, v_2)$, and $u_{R_\beta(1)} < u_2 < u_*$.

Starting from (u_+, v_+), we construct $S_2^{(\mathrm{II})}(u_+, v_+)$. It is clear that $(u_2, v_2) \in S_2^{(\mathrm{II})}(u_+, v_+)$. We join (u_2, v_2) and $R_\beta(1)$ by a 2-rarefaction wave then since $(u_2, v_2) \in R_2(R_\beta(1))$. At last, we construct $S_2^{(\mathrm{II})}(R_\beta(1))$. It can be shown that $(u_1, v_1) \in S_2^{(\mathrm{II})}(R_\beta(1))$ with $\sigma_2 = \lambda_2(R_\beta(1))$. Starting from (u_1, v_1), we carry out the same procedure as before. Therefore, we can construct the correspoding type II solution with the same configuration as shown in Fig. 4.5 and Fig. 4.6, resectively.

When (u_+, v_+) is located in between the curves $R_\beta(1)$ and $u = u_*$ (see Fig. 3.2), there are also two kinds of wave patterns for the type I solution which can be described in the same way as above without the last wave S_2 and replacing (u_2, v_2) by (u_+, v_+). We omit the detail.

When (u_+, v_+) is located in between the curves $\bar{W}_1(u_-, v_-)$ and $R_\beta(1)$, the wave patterns from (u_1, v_1) are

$$(u_1, v_1) \xrightarrow{S_2} (u_+, v_+)$$

for the type I solution. It is easy to show that $(u_1, v_1) \in S_2^{(\mathrm{II})}(u_+, v_+)$, which, together with the same discussion, starting from (u_2, v_2) and ending at (u_-, v_-), gives us the corresponding type II solution with the same configuration as the type I solution, respectively.

When (u_+, v_+) is located on the left of the curve $\bar{W}_1(u_-, v_-)$, the discussion is the same as in Case 1.

Case 3. $(u_+, v_+) \in G_3$.

When (u_+, v_+) is located on the right of the curve $L_\beta(1)$, the wave patterns for the type I solution are

$$(u_-, v_-) \xrightarrow{S_1 \text{ or } R_1} (u_1, v_1) \xrightarrow{S_2} (u_+, v_+),$$

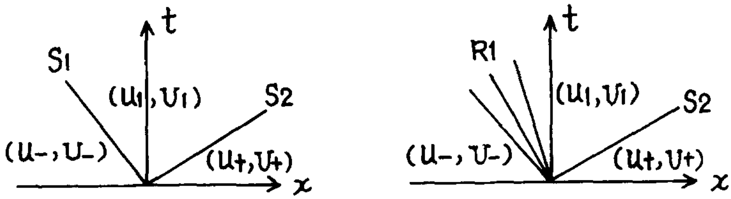

FIGURE 4.7

as shown in Fig. 4.7. It is easy to see that $(u_1, v_1) \in S_2^{(\mathrm{II})}(u_+, v_+)$ since $u_\beta < u_1 < u_*$ and $u_+ > u_{L_\beta(1)}$. Starting from (u_1, v_1), it is obvious that $(u_-, v_-) \in S_1^{(\mathrm{II})}(u_1, v_1)$ when $(u_1, v_1) \in S_1(u_-, v_-)$ while $(u_-, v_-) \in R_1^{(\mathrm{II})}(u_1, v_1)$ when $(u_1, v_1) \in R_1(u_-, v_-)$. Since both u_- and u_1 are in the interval (u_β, u_*), therefore, we can construct the type II solution with the same configuration, shown in Fig. 4.7.

When (u_+, v_+) is located in between the curves $u = u_*$ and $L_\beta(1)$, the wave patterns for the type I solution are

$$(u_-, v_-) \xrightarrow{S_1 \text{ or } R_1} (u_1, v_1) \xrightarrow{R_2} (u_2, v_2) \xrightarrow[\text{with } \sigma_2 = \lambda_2(u_2)]{S_2} (u_+, v_+),$$

as shown in Fig. 4.8, where $(u_+, v_+) = L_\beta(u_2, v_2)$, $u_1 < u_2 < u_*$.

Due to the locations of u_2 and u_+, it is known that $(u_2, v_2) \in S_2^{(\mathrm{II})}(u_+, v_+)$ and $(u_1, v_1) \in R_2^{(\mathrm{II})}(u_2, v_2)$, which, together with the same discussion concerning the connection of (u_1, v_1) to (u_-, v_-), shows that the corresponding type II solution with the same configuration as in Fig. 4.8 can be constructed.

When (u_+, v_+) is located in between the curves $\bar{w}_1(u_-, v_-)$ and $u = u_*$, the wave patterns for the type I solution are the same as above without the last wave S_2, replacing (u_2, v_2) by (u_+, v_+). We omit the discussion.

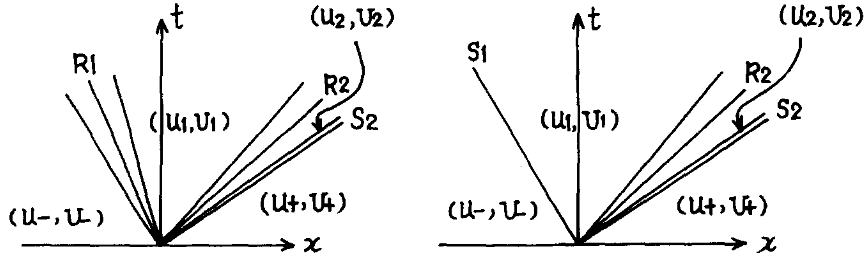

FIGURE 4.8

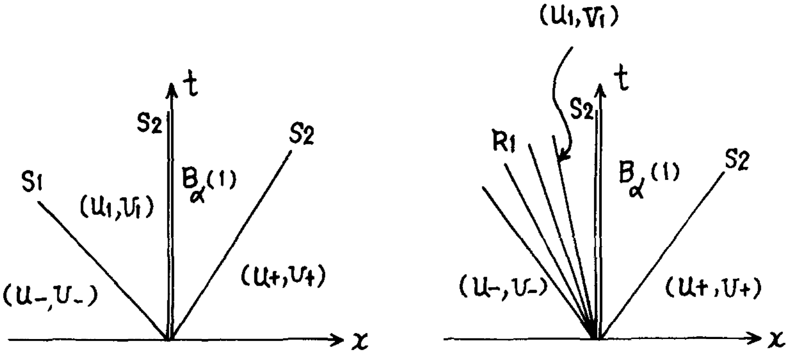

FIG. 4.9. The set consisting of the t-axis and a ray parallel to the t-axis denotes the contact discontinuity with speed 0.

When (u_+, v_+) is located in between the curves $B_e(1)$ and $\tilde{W}_1(u_-, v_-)$ (see Fig. 3.2), the wave patterns for the type I solution are

$$(u_-, v_-) \xrightarrow{S_1 \text{ or } R_1} (u_1, v_1) \xrightarrow{S_2} (u_+, v_+).$$

It is clear that $(u_2, v_2) \in S_2^{(\mathrm{II})}(u_+, v_+)$ since $u_{B_e(1)} < u_+ < u_1$. The corresponding type II solution with the same configuration can then be constructed.

When (u_+, v_+) is located in between the curves $B_\alpha(1)$ and $B_e(1)$, the wave patterns for the type I solution are

$$(u_-, v_-) \xrightarrow{S_1 \text{ or } R_1} (u_1, v_1) \xrightarrow[\text{with } \sigma_2 = 0]{S_2} (B_\alpha(1)) \xrightarrow{S_2} (u_+, v_+),$$

as shown in Fig. 4.9. It is easy to show that $B_\alpha(1) \in S_2^{(\mathrm{II})}(u_+, v_+)$, $(u_1, v_1) \in S_1^{(\mathrm{II})}(B_\alpha(1))$ with $\sigma_1 = 0$, which, together with the same discussion concerning the connection between (u_1, v_1) and (u_-, v_-) as before, implies that the corresponding type II solutions with the same configuration as in Fig. 4.9 can be constructed.

When (u_+, v_+) is located on the left of the curve $B_\alpha(1)$ (see (Fig. 3.2), the wave patterns for the type I solution are

$$(u_-, v_-) \xrightarrow{S_1 \text{ or } R_1} (u_1, v_1) \xrightarrow[\text{with } \sigma_2 = 0]{S_2} B_\alpha(1) \xrightarrow{R_2} (u_+, v_+),$$

as shown in Fig. 4.10. Obviously $B_\alpha(1) \in R_2^{(\mathrm{II})}(u_+, v_+)$, and the left part can be carried out in the same way as before.

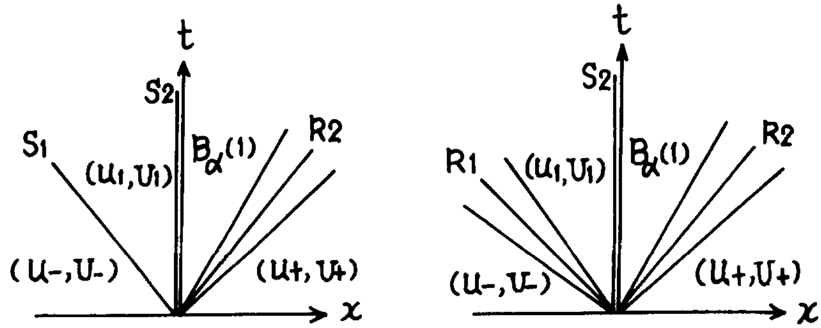

FIGURE 4.10

Case 4. $(u_+, v_+) \in G_4$.

When (u_+, v_+) is located on the right of the curve $\bar{W}_1(u_-, v_-)$, the wave patterns for the type I solution are either

$$(u_-, v_-) \xrightarrow{S_1} (u_1, v_1) \xrightarrow{S_2} (u_+, v_+),$$

as shown in Fig. 4.11, or

$$(u_-, v_-) \xrightarrow[\text{with } \sigma_1 = \lambda_1(R_\beta(-))]{S_1} (R_\beta(-)) \xrightarrow{R_1} (u_1, v_1) \xrightarrow{S_2} (u_+, v_+),$$

as shown in Fig. 4.12. It can be easily shown that $(u_1, v_1) \in S_2^{(\mathrm{II})}(u_+, v_+)$, $(u_-, v_-) \in S_1^{(\mathrm{II})}(u_1, v_1)$ for the first case while $(u_-, v_-) \in S_1^{(\mathrm{II})}(R_\beta(-))$ with $\sigma_1 = \lambda_1(R_\beta(-))$ for the second case. Therefore, the corresponding type II solution can be constructed which has the same configuration as shown in Fig. 4.11 or Fig. 4.12, respectively.

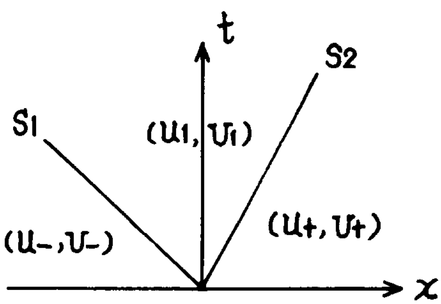

FIGURE 4.11

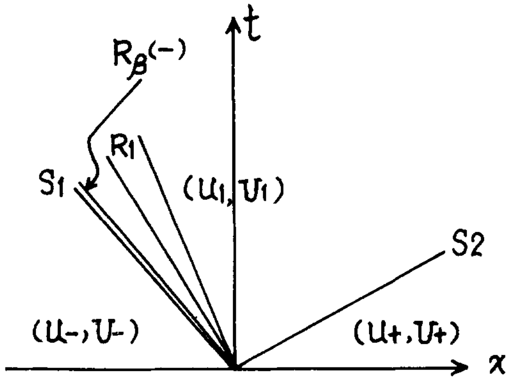

FIGURE 4.12

When (u_+, v_+) is located in between the curves $u = u_*$ and $W_1(u_-, v_-)$, the wave patterns for the type I solution are either

$$(u_-, v_-) \xrightarrow{S_1} (u_1, v_1) \xrightarrow{R_2} (u_+, v_+)$$

or

$$(u_-, v_-) \xrightarrow[\text{with } \sigma_1 = \lambda_1(R_\beta(-))]{S_1} (R^\beta(-)) \xrightarrow{R_2} (u_1, v_1) \xrightarrow{R_2} (u_+, v_+).$$

Obviously, $(u_1, v_1) \in R^{(\mathrm{II})}(u_+, v_+)$, the left part of the discussion, can be carried out in the same way as before, and we omit the detail.

When (u_+, v_+) is located in between the curves $L_e(1)$ and $u = u_*$, the wave patterns for the type I solution from (u_-, v_-) to (u_1, v_1) are the same as before while from (u_1, v_1) to (u_+, v_+) they are

$$(u_1, v_1) \xrightarrow{R_2} (u_2, v_2) \xrightarrow[\text{with } \sigma_2 = \lambda_2(u_2)]{S_2} (u_+, v_+),$$

where $(u_+, v_+) = L_e(u_2, v_2)$. Due to $(u_2, v_2) \in S_2^{(\mathrm{II})}(u_+, v_+)$ with $\sigma_2 = \lambda_2(u_2)$ and $(u_1, v_1) \in R_2^{(\mathrm{II})}(u_2, v_2)$, it follows that the corresponding type II solution with the same configuration as the type I solution can be constructed.

When (u_+, v_+) is located in between the curves $B_e(1)$ and $L_e(1)$, the wave patterns for the type I solution from (u_1, v_1) to (u_+, v_+) are

$$(u_1, v_1) \xrightarrow{S_2} (u_+, v_+).$$

Since $(u_1, v_1) \in S_2^{(\mathrm{II})}(u_+, v_+)$, the corresponding type II solution with the same configuration as the type I solution, can be constructed.

When (u_+, v_+) is located in between the curves $B_\alpha(1)$ and $B_e(1)$, the wave patterns for the type I solution from (u_1, v_1) to (u_+, v_+) are

$$(u_1, v_1) \xrightarrow[\text{with } \sigma_2 = 0]{S_2} B_\alpha(1) \xrightarrow{S_2} (u_+, v_+).$$

Due to $B_\alpha(1) \in S_2^{(\mathrm{II})}(u_+, v_+)$ and $(u_1, v_1) \in S_1^{(\mathrm{II})}(B_\alpha(1))$ with $\sigma_1 = 0$, we are able to construct the type II solution which has the same configuration as the corresponding type I solution.

When (u_+, v_+) is located on the left of $B_\alpha(1)$, the wave patterns for the type I solution from (u_1, v_1) to (u_+, v_+) are

$$(u_1, v_1) \xrightarrow[\text{with } \sigma_2 = 0]{S_2} B_\alpha(1) \xrightarrow{R_2} (u_+, v_+).$$

Since $B_\alpha(1) \in R_2^{(\mathrm{II})}(u_+, v_+)$ and $(u_1, v_1) \in S_1^{(\mathrm{II})}(B_\alpha(1))$ with $\sigma_1 = 0$, the corresponding type II solution can be constructed with the same configuration as those type I solutions.

Case 5. $(u_+, v_+) \in G_5$.

The wave patterns for the type I solution are

$$(u_-, v_-) \xrightarrow[\text{with } \sigma_1 = \lambda_1(R_\beta(-))]{S_1} (R_\beta(-)) \xrightarrow{R_1} (u_1, v_1) \xrightarrow{S_2} (u_+, v_+)$$

when (u_+, v_+) is located on the right of $R_1(R_\beta(-))$ (see Fig. 3.2);

$$(u_-, v_-) \xrightarrow[\text{with } \sigma_1 = \lambda_1(R_\beta(-))]{S_1} (R_\beta(-)) \xrightarrow{R_2} (u_1, v_1) \xrightarrow{R_2} (u_+, v_+)$$

when (u_+, v_+) is located in between $u = u_*$ and $R_1(R_\beta(-))$;

$$(u_-, v_-) \xrightarrow[\text{with } \sigma_1 = \lambda_1(R_\beta(-))]{S_1} (R_\beta(-)) \xrightarrow{R_1} (u_1, v_1) \xrightarrow{R_2} (u_2, v_2)$$

$$\xrightarrow[\text{with } \sigma_2 = \lambda_2(u_2)]{S_2} (u_+, v_+)$$

when (u_+, v_+) is located in between the curves $L_e(1)$ and $u = u_*$, where $(u_+, v_+) = L_e(u_2, v_2)$;

$$(u_-, v_-) \xrightarrow[\text{with } \sigma_1 = \lambda_1(R_\beta(-))]{S_1} (R_\beta(-)) \xrightarrow{R_1} (u_1, v_1) \xrightarrow{R_2} (u_+, v_+)$$

when (u_+, v_+) is located in between the curves $R_\alpha(1)$ and $L_e(1)$;

$$(u_-, v_-) \xrightarrow[\text{with } \sigma_1 = \lambda_1(R_\beta(-))]{S_1} (R_\beta(-)) \xrightarrow{R_1} (u_1, v_1)$$

$$\xrightarrow[\text{with } \sigma_2 = \lambda_2(R_\alpha(1))]{S_2} (R_\alpha(1)) \xrightarrow{R_2} (u_+, v_+)$$

when (u_+, v_+) is located on the left of $R_\alpha(1)$ (see Fig. 3.2).

No matter which cases, it can be shown, similarly as before, that the type II solution with the same configuration can be constructed respectively.

Case 6. $(u_+, v_+) \in G_6$.

The wave patterns for the type I solution are

$$(u_-, v_-) \xrightarrow[\text{with } \sigma_1 = \lambda_1(R_\beta(-))]{S_1} (R_\beta(-)) \xrightarrow{R_1} (u_1, v_1) \xrightarrow{S_2} (u_+, v_+)$$

when (u_+, v_+) is located on the right of $R_1(R_\beta(-))$ (see Fig. 3.2);

$$(u_-, v_-) \xrightarrow[\text{with } \sigma_1 = \lambda_1(R_\beta(-))]{S_1} (R_\beta(-)) \xrightarrow{R_1} (u_1, v_1) \xrightarrow{R_2} (u_+, v_+)$$

when (u_+, v_+) is located in between $u = u_*$ and $R_1(R_\beta(-))$;

$$(u_-, v_-) \xrightarrow[\text{with } \sigma_1 = \lambda_1(R_\beta(-))]{S_1} (R_\beta(-)) \xrightarrow{R_1} (u_1, v_1) \xrightarrow{R_2} (u_2, v_2)$$

$$\xrightarrow[\text{with } \sigma_2 = \lambda_2(u_2)]{S_2} (u_+, v_+)$$

when (u_+, v_+) is located in between $u = \tilde{u}$ and $u = u_*$;

$$(u_-, v_-) \xrightarrow[\text{with } \sigma_1 = \lambda_1(R_\beta(-))]{S_1} (R_\beta(-)) \xrightarrow{R_1} (u_1, v_1) \xrightarrow{R_2} (u_2, v_2)$$

$$\xrightarrow[\text{with } \sigma_2 = \lambda_2(u_2)]{S_2} (\hat{u}_2, \hat{v}_2) \xrightarrow{R_2} (u_+, v_+), \quad \text{where } u_2 = \tilde{\tilde{u}}, \hat{u}_2 = \tilde{u}.$$

It can be shown easily that $(\hat{u}_2, \hat{v}_2) \in R_2^{(\mathrm{II})}(u_+, v_+)$, $(u_2, v_2) \in S_2^{(\mathrm{II})}(\hat{u}_2, \hat{v}_2)$ with $\sigma_2 = \lambda_2(u_2) = \lambda_2(\hat{u}_2)$ and $(u_1, v_1) \in R_2^{(\mathrm{II})}(u_2, v_2)$. This, with the facts $(u_1, v_1) \in R_1(R_\beta(-))$ and $(u_-, v_-) \in S_1^{(\mathrm{II})}(R_\beta(-))$ with $\sigma_1 = \lambda_1(R_\beta(-))$, implies that the type II solution can be constructed with the same configuration as the corresponding type I solution for the last situation. The other situations can be treated easily as well.

We end up with

THEOREM 4.1. *For any given Riemann data, the unique admissible type* I *weak solution and the unique admissible type* II *weak solution of the Riemann problem* (2.1), (3.1) *are identical.*

This shows that the generalized shock (E) criterion is a suitable admissibility criterion for system (1.2) of mixed type. Moreover, we may introduce the admissible type III weak solution in which any discontinuity of the first kind satisfies the admissibility criterion type I while any discontinuity of the second kind satisfies the admissibility criterion type II.

It can be shown that the admissible type III weak solution is the same as the type I and type II for given Riemann data belonging to the same phase [HS2]. This gives the approach for nonisothermal motion to the mixed type system of conservation laws, [HS2].

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